CPS 173
Voting and social choice

Vincent Conitzer
conitzer@cs.duke.edu
Voting over alternatives

voting rule (mechanism) determines winner based on votes

• Can vote over other things too
  – Where to go for dinner tonight, other joint plans, …
Voting (rank aggregation)

- Set of $m$ candidates (aka. alternatives, outcomes)
- $n$ voters; each voter ranks all the candidates
  - E.g., for set of candidates $\{a, b, c, d\}$, one possible vote is $b > a > d > c$
  - Submitted ranking is called a vote

- A voting rule takes as input a vector of votes (submitted by the voters), and as output produces either:
  - the winning candidate, or
  - an aggregate ranking of all candidates

- Can vote over just about anything
  - political representatives, award nominees, where to go for dinner tonight, joint plans, allocations of tasks/resources, …
  - Also can consider other applications: e.g., aggregating search engines’ rankings into a single ranking
Example voting rules

- **Scoring rules** are defined by a vector \((a_1, a_2, \ldots, a_m)\); being ranked \(i\)th in a vote gives the candidate \(a_i\) points
  - **Plurality** is defined by \((1, 0, 0, \ldots, 0)\) (winner is candidate that is ranked first most often)
  - **Veto** (or anti-plurality) is defined by \((1, 1, \ldots, 1, 0)\) (winner is candidate that is ranked last the least often)
  - **Borda** is defined by \((m-1, m-2, \ldots, 0)\)

- **Plurality with (2-candidate) runoff**: top two candidates in terms of plurality score proceed to runoff; whichever is ranked higher than the other by more voters, wins

- **Single Transferable Vote** (STV, aka. Instant Runoff): candidate with lowest plurality score drops out; if you voted for that candidate, your vote transfers to the next (live) candidate on your list; repeat until one candidate remains

- Similar runoffs can be defined for rules other than plurality
Pairwise elections

two votes prefer Obama to McCain

two votes prefer Obama to Nader

two votes prefer Nader to McCain

>   >   >

>   >   >

>   >   >

>   >   >
Condorcet cycles

two votes prefer McCain to Obama

two votes prefer Obama to Nader

two votes prefer Nader to McCain

“weird” preferences
Voting rules based on pairwise elections

- **Copeland**: candidate gets two points for each pairwise election it wins, one point for each pairwise election it ties

- **Maximin (aka. Simpson)**: candidate whose worst pairwise result is the best wins

- **Slater**: create an overall ranking of the candidates that is inconsistent with as few pairwise elections as possible
  - NP-hard!

- **Cup/pairwise elimination**: pair candidates, losers of pairwise elections drop out, repeat
Even more voting rules…

- **Kemeny**: create an overall ranking of the candidates that has as few *disagreements* as possible (where a disagreement is with a vote on a pair of candidates)
  - NP-hard!
- **Bucklin**: start with $k=1$ and increase $k$ gradually until some candidate is among the top $k$ candidates in more than half the votes; that candidate wins
- **Approval** (not a ranking-based rule): every voter labels each candidate as approved or disapproved, candidate with the most approvals wins
Pairwise election graphs

- **Pairwise election** between $a$ and $b$: compare how often $a$ is ranked above $b$ vs. how often $b$ is ranked above $a$
- Graph representation: edge from winner to loser (no edge if tie), weight = margin of victory
- E.g., for votes $a > b > c > d$, $c > a > d > b$ this gives

\[
\begin{align*}
\text{a} & \rightarrow \text{b} \\
\text{d} & \rightarrow \text{c} \\
\text{d} & \rightarrow \text{b} \\
\text{c} & \rightarrow \text{a}
\end{align*}
\]
Kemeny on pairwise election graphs

- Final ranking = acyclic tournament graph
  - Edge (a, b) means a ranked above b
  - Acyclic = no cycles, tournament = edge between every pair
- Kemeny ranking seeks to minimize the total weight of the inverted edges

**pairwise election graph**

**Kemeny ranking**

\[(b > d > c > a)\]
Slater on pairwise election graphs

- Final ranking = acyclic tournament graph
- Slater ranking seeks to minimize the number of inverted edges

pairwise election graph

Slater ranking

(a > b > d > c)
An integer program for computing Kemeny/Slater rankings

$y_{(a, b)}$ is 1 if $a$ is ranked below $b$, 0 otherwise

$w_{(a, b)}$ is the weight on edge $(a, b)$ (if it exists)

in the case of Slater, weights are always 1

minimize: $\sum_{e \in E} w_e y_e$

subject to:

for all $a, b \in V$, $y_{(a, b)} + y_{(b, a)} = 1$

for all $a, b, c \in V$, $y_{(a, b)} + y_{(b, c)} + y_{(c, a)} \geq 1$
Choosing a rule

• How do we choose a rule from all of these rules?
• How do we know that there does not exist another, “perfect” rule?
• Let us look at some criteria that we would like our voting rule to satisfy
Condorcet criterion

- A candidate is the Condorcet winner if it wins all of its pairwise elections
- Does not always exist...
- … but the Condorcet criterion says that if it does exist, it should win

- Many rules do not satisfy this
- E.g. for plurality:
  - b > a > c > d
  - c > a > b > d
  - d > a > b > c
- a is the Condorcet winner, but it does not win under plurality
Majority criterion

• If a candidate is ranked first by most votes, that candidate should win
  – Relationship to Condorcet criterion?

• Some rules do not even satisfy this

• E.g. Borda:
  – a > b > c > d > e
  – a > b > c > d > e
  – c > b > d > e > a

• a is the majority winner, but it does not win under Borda
Monotonicity criteria

• Informally, monotonicity means that “ranking a candidate higher should help that candidate,” but there are multiple nonequivalent definitions

• A \textit{weak} monotonicity requirement: if
  – candidate \( w \) wins for the current votes,
  – we then improve the position of \( w \) in some of the votes and leave everything else the same,

then \( w \) should still win.

• E.g., STV does not satisfy this:
  – 7 votes \( b > c > a \)
  – 7 votes \( a > b > c \)
  – 6 votes \( c > a > b \)

• \( c \) drops out first, its votes transfer to \( a \), \( a \) wins

• But if 2 votes \( b > c > a \) change to \( a > b > c \), \( b \) drops out first, its 5 votes transfer to \( c \), and \( c \) wins
Monotonicity criteria...

• A **strong** monotonicity requirement: if
  – candidate w wins for the current votes,
  – we then change the votes in such a way that for each vote, if a candidate c was ranked below w originally, c is still ranked below w in the new vote

then w should still win.

• Note the other candidates can jump around in the vote, as long as they don’t jump ahead of w

• None of our rules satisfy this
Independence of irrelevant alternatives

- Independence of irrelevant alternatives criterion: if
  – the rule ranks a above b for the current votes,
  – we then change the votes but do not change which is ahead between a and b in each vote
then a should still be ranked ahead of b.
- None of our rules satisfy this
Arrow’s impossibility theorem [1951]

- Suppose there are at least 3 candidates
- Then there exists no rule that is simultaneously:
  - Pareto efficient (if all votes rank a above b, then the rule ranks a above b),
  - nondictatorial (there does not exist a voter such that the rule simply always copies that voter’s ranking), and
  - independent of irrelevant alternatives
Muller-Satterthwaite impossibility theorem [1977]

- Suppose there are at least 3 candidates
- Then there exists no rule that simultaneously:
  - satisfies *unanimity* (if all votes rank a first, then a should win),
  - is *nondictatorial* (there does not exist a voter such that the rule simply always selects that voter’s first candidate as the winner), and
  - is *monotone* (in the strong sense).
Manipulability

- Sometimes, a voter is better off revealing her preferences insincerely, aka. manipulating

- E.g. plurality
  - Suppose a voter prefers $a > b > c$
  - Also suppose she knows that the other votes are
    - 2 times $b > c > a$
    - 2 times $c > a > b$
  - Voting truthfully will lead to a tie between $b$ and $c$
  - She would be better off voting e.g. $b > a > c$, guaranteeing $b$ wins

- All our rules are (sometimes) manipulable
Gibbard-Satterthwaite impossibility theorem

• Suppose there are at least 3 candidates
• There exists no rule that is simultaneously:
  – onto (for every candidate, there are some votes that would make that candidate win),
  – nondictatorial (there does not exist a voter such that the rule simply always selects that voter’s first candidate as the winner), and
  – nonmanipulable
Single-peaked preferences

• Suppose candidates are ordered on a line
• Every voter prefers candidates that are closer to her most preferred candidate
• Let every voter report only her most preferred candidate ("peak")
• Choose the median voter’s peak as the winner
  – This will also be the Condorcet winner
• Nonmanipulable!

Impossible results do not necessarily hold when the space of preferences is restricted
Some computational issues in social choice

• Sometimes **computing the winner/aggregate ranking** is hard
  – E.g. for Kemeny and Slater rules this is NP-hard
• For some rules (e.g., STV), computing a successful *manipulation* is NP-hard
  – Manipulation being hard is a **good** thing (circumventing Gibbard-Satterthwaite?)… But would like something stronger than NP-hardness
  – Also: work on the complexity of **controlling** the outcome of an election by influencing the list of candidates/schedule of the Cup rule/etc.

• **Preference elicitation:**
  – We may not want to force each voter to rank **all** candidates;
  – Rather, we can selectively query voters for parts of their ranking, according to some algorithm, to obtain a good aggregate outcome

• **Combinatorial alternative spaces:**
  – Suppose there are multiple interrelated issues that each need a decision
  – Exponentially sized alternative spaces

• **Different models** such as ranking webpages (pages “vote” on each other by linking)