Homework 5 : Numerical Solutions to ODEs and PDEs

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1. (15) Consider numerical solution for the single-cell, one action potential model, i.e., the initial value problem with an ordinary differential equation (ODE)

$$\frac{du}{dt} = f(u) = u(\tau - u)(u-1), \quad \tau \in (0,1)$$

$$\text{i.c.} \qquad u(0) = u_0.$$

   (a) experiment with the provided code on two different values of the threshold τ, ten different initial values and at least two different values of the time step;

   (b) make changes in the code to use a different integration method, and make experimental comparisons, observation and comments.

2. (15) Consider numerical solutions for the single-cell model with double action potentials, one fast and one slow, i.e., the initial value problem with two ODEs

$$\frac{du}{dt} = f(u) = u(\tau - u)(1-u) - w + Sim(t), \quad \tau \in (0,1)$$

$$\frac{dw}{dt} = \alpha u - \beta w$$

$$\text{i.c.} \qquad [u(0), w(0)] = [u_0, w_0]$$

   (a) experiment with the provided code and two sets of model parameters α, β and τ, using the initial value vectors of your design or choice.

   (b) make changes in the code to use an implicit integration method or a partially implicit method, and make experimental comparisons and comments.

3. (30) Consider numerical solutions for the following cell ring model with two interaction potentials within each cell and interaction via the fast action potential across the cells, i.e., a system of partial differential equations (PDEs) with initial values and boundary values for $[u(x,t), w(x,t)]$,

$$\begin{aligned}
\frac{\partial u}{\partial t} &= f(u) = \Delta u + u(\tau - u)(1-u) - w, \quad x \in (L, R) \\
\frac{dw}{dt} &= \alpha u - \beta w \\
\text{i.c.} \quad & u(x,0) = u_0(x); \qquad w(x,0) = w_0(x); \\
\text{b.c.} \quad & u(L,t) = u(R,t); \qquad w(L,t) = w(R,t);
\end{aligned}$$

(a) experiment with the provided code, two sets of model parameters α, β and τ, and provided initial values and boundary values, but with two spatial discretization scales and two different time steps (four space-time discretization combinations).

(b) make comments on the difference between the single cell model and multiple-cells model, and comment on how PDEs are converted into ODEs via spatial discretization.

(c) make changes in the code to use an implicit integration method, and make experimental comparisons and comments.

4. (40) Extend the work on Problem 3 to the 2D case for numerical solution of $[u(x,y,t), w(x,y,t)]$ at discretized space-time locations.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= f(u) = \Delta u + u(\tau - u)(1-u) - w, \quad (x,y) \in \Omega \\
\frac{dv}{dt} &= \alpha u - \beta v \\
\text{i.c.} \quad u(x,y,0) &= u_0(x,y); \quad w(x,y,0) = w_0(x,y); \quad (x,y) \in \Omega \\
\text{b.c.} \quad u(x,y,t) &= u_{\partial\Omega}(t); \qquad w(x,y,t) = w_{\partial\Omega}(t), \quad (x,y) \in \partial\Omega
\end{aligned}$$

Adaptive Quadratures

1. Let Q_1 be the trapezoidal rule on interval Δ of length $h < 1$. Let Q_2 be the trapezoidal method applied to the two half intervals of Δ. Find a linear combination $Q_3 = \alpha Q_1 + \beta Q_2$ such that Q_3 is of higher precision in polynomial degree. In addition, use $Q_3(f, \Delta)$ to estimate the dominant-degree terms in the errors of $Q_1(f)$ and $Q_2(f)$, respectively.

2. Make a similar analysis with the Simpson's rule.