Today’s topics

Game Theory
Normal-form games
Dominating strategies
Nash equilibria

Acknowledgements
Vincent Conitzer, Michael Kearns

Rock-paper-scissors

Column player aka. player 2
(simultaneously) chooses a column

Row player aka. player 1
chooses a row

A row or column is called an action or (pure) strategy

Row player’s utility is always listed first, column player’s second

Zero-sum game: the utilities in each entry sum to 0 (or a constant)
Three-player game would be a 3D table with 3 utilities per entry, etc.

Matching pennies (~penalty kick)

Two players drive cars towards each other
• If one player goes straight, that player wins
• If both go straight, they both die

“Chicken”
**RPS – Seinfeld variant**

MICKEY: All right, rock beats paper! (Mickey smacks Kramer's hand for losing)
KRAMER: I thought paper covered rock.
MICKEY: Nah, rock flies right through paper.
KRAMER: What beats rock?
MICKEY: (looks at hand) Nothing beats rock.

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**Dominance**

- **Player i’s strategy s_i strictly dominates s_i’ if**
  - for any s_{-i} \( P_i(s_i, s_{-i}) > P_i(s_i', s_{-i}) \)
- **s_i weakly dominates s_i’ if**
  - for any s_{-i} \( P_i(s_i, s_{-i}) \geq P_i(s_i', s_{-i}); \) and
  - for some s_{-i} \( P_i(s_i, s_{-i}) > P_i(s_i', s_{-i}) \)

- \(-i = “the player(s) other than i”\)

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**Prisoner’s Dilemma**

- Pair of criminals has been caught
- District attorney has evidence to convict them of a minor crime (4 years in jail); knows that they committed a major crime together (10 years in jail) but cannot prove it
- Offers them a deal:
  - If both confess to the major crime, they each get a 6 year reduction
  - If only one confesses, that one gets 9 year reduction

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<tr>
<td>don’t confess</td>
<td>don’t confess</td>
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**“Should I buy an SUV?”**

- **purchasing cost**
  - cost: 5
  - cost: 3
  - cost: 8
  - cost: 5
- **accident cost**
  - cost: 5
  - cost: 2
  - cost: 5
  - cost: 5

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Mixed strategies

- **Mixed strategy** for player $i$ = probability distribution over player $i$’s (pure) strategies
- E.g., 1/3, 1/3, 1/3
- Example of dominance by a mixed strategy:

```
1/2 3, 0 0, 0
1/2 0, 0 3, 0
```

Usage:
- $\sigma_i$ denotes a mixed strategy,
- $s_i$ denotes a pure strategy

**Example:**
- Mixed strategy for player $i$ = probability distribution over player $i$’s (pure) strategies
- E.g., $1/3$, $1/3$, $1/3$

John Forbes Nash

- A mixed strategy for a player is a distribution on their available actions
  - e.g. 1/3 rock, 1/3 paper, 1/3 scissors
- Joint mixed strategy for N players:
  - a distribution for each player (possibly different)
  - assume everyone knows all the distributions
  - but the “coin flips” used to select from player $i$’s distribution known only to $i$
    - “private randomness”
    - so only player $i$ knows their actual choice of action
    - can people randomize? (more later)
- Joint mixed strategy is an equilibrium if:
  - for every player $i$, their distribution is a best response to all the others
    - i.e. cannot get higher (average or expected) payoff by changing distribution
    - only consider unilateral deviations by each player!
  - Nash 1950: every game has a mixed strategy equilibrium!
  - no matter how many rows and columns there are
  - in fact, no matter how many players there are
- Thus known as a Nash equilibrium
- A major reason for Nash’s Nobel Prize in economics

Facts about Nash Equilibria

- While there is always at least one, there might be many
  - zero-sum games: all equilibria give the same payoffs to each player
  - non zero-sum: different equilibria may give different payoffs!
- Equilibrium is a static notion
  - does not suggest how players might learn to play equilibrium
  - does not suggest how we might choose among multiple equilibria
- Nash equilibrium is a strictly competitive notion
  - players cannot have “pre-play communication”
  - bargains, side payments, threats, collusions, etc. not allowed
- Computing Nash equilibria for large games is difficult

Nash equilibria of “chicken”...

<table>
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- Is there a Nash equilibrium that uses mixed strategies? Say, where player 1 uses a mixed strategy?
- Recall: if a mixed strategy is a best response, then all of the pure strategies that it randomizes over must also be best responses
- So we need to make player 1 indifferent between D and S
- Player 1’s utility for playing $D = -p_s$
- Player 1’s utility for playing $S = p_D - 5p_s = 1 - 6p_s$
- So we need $-p_s = 1 - 6p_s$, which means $p_s = 1/5$
- Then, player 2 needs to be indifferent as well
- Mixed-strategy Nash equilibrium: $((4/5 D, 1/5 S), (4/5 D, 1/5 S))$
  - People may die! Expected utility -1/5 for each player