Slides from Kevin Wayne on Union-Find and Percolotion

Subtext of today's lecture (and this course)

Steps to developing a usable algorithm.

- Model the problem.
- Find an algorithm to solve it.
- Fast enough? Fits in memory?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.

The scientific method.

Mathematical analysis.

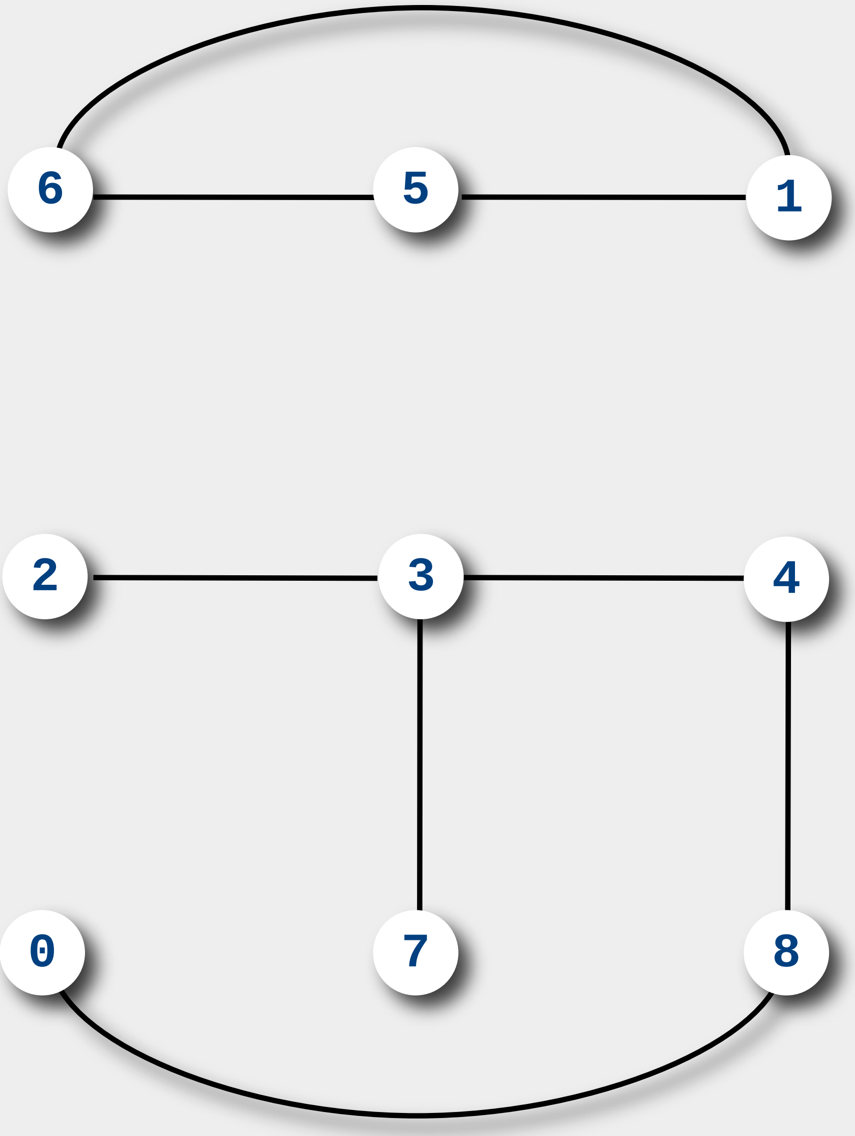
- dynamic connectivity
- quick find
- quick union
- improvements
- applications

Given a set of objects

- Union: connect two objects.
- Connected: is there a path connecting the two objects?

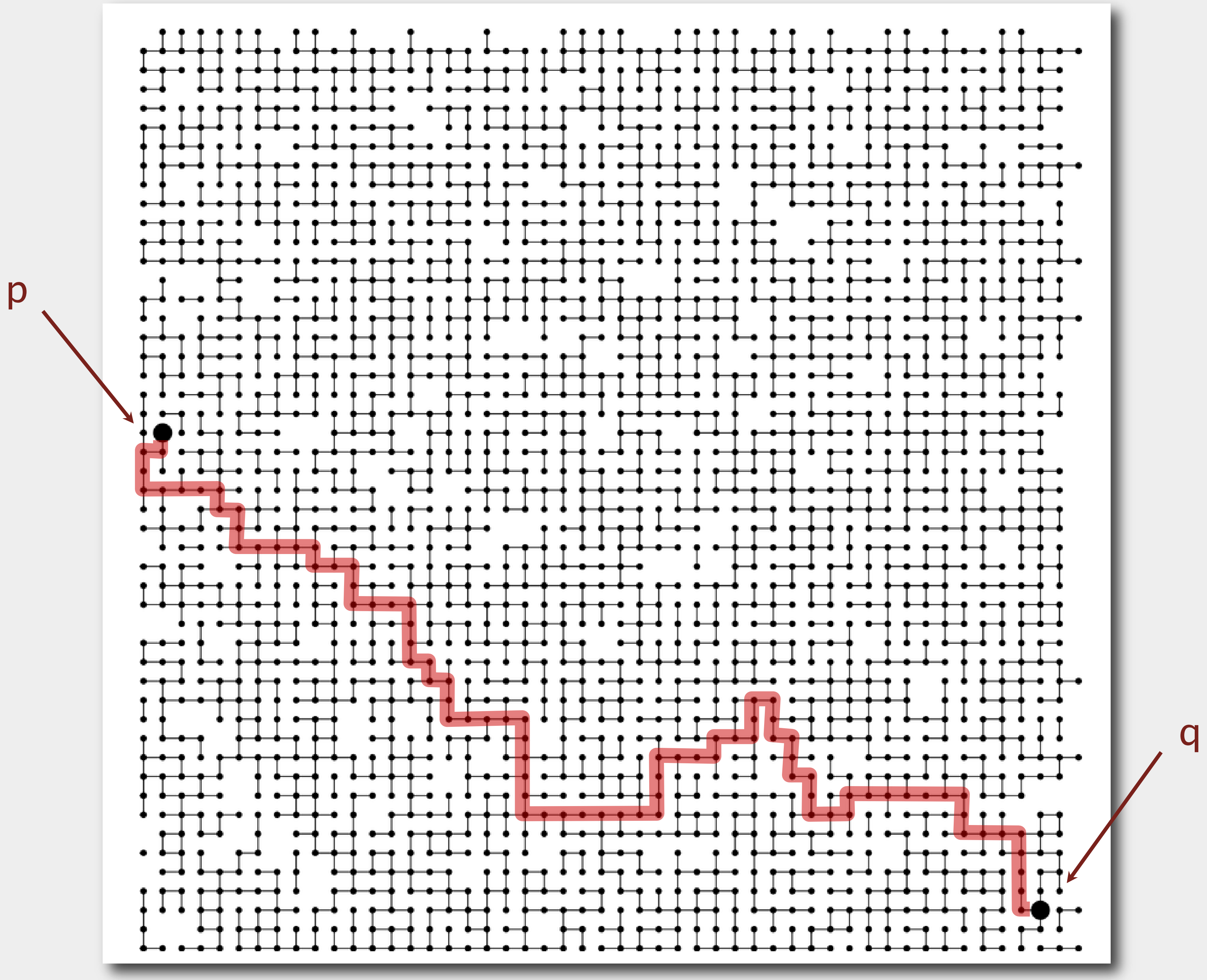
more difficult problem: find the path

```
union(3, 4)
union(8, 0)
union(2, 3)
union(5, 6)
connected(0, 2)     no
connected(2, 4)     yes
union(5, 1)
union(7, 3)
union(1, 6)
union(4, 8)
connected(0, 2)     yes
connected(2, 4)     yes
```



Connectivity example

Q. Is there a path from p to q?



A. Yes.

Modeling the objects

Dynamic connectivity applications involve manipulating objects of all types.

- Pixels in a digital photo.
- Computers in a network.
- Variable names in Fortran.
- Friends in a social network.
- Transistors in a computer chip.
- Elements in a mathematical set.
- Metallic sites in a composite system.

When programming, convenient to name sites 0 to N-1.

- Use integers as array index.
- Suppress details not relevant to union-find.

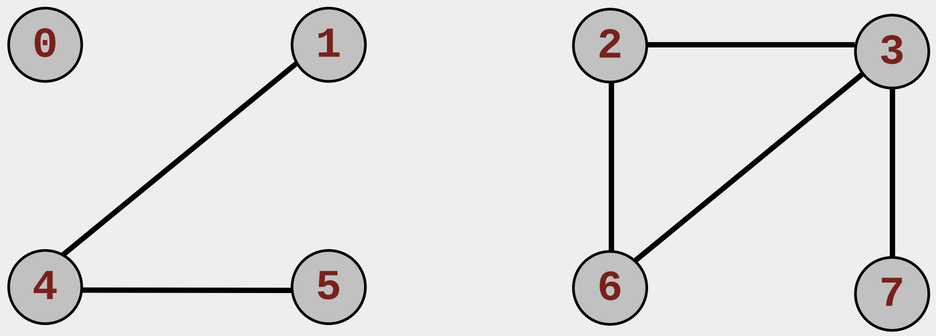
can use symbol table to translate from site names to integers: stay tuned (Chapter 3)

Modeling the connections

We assume "is connected to" is an equivalence relation:

- Reflexive: p is connected to p.
- Symmetric: if p is connected to q, then q is connected to p.
- Transitive: if p is connected to q and q is connected to r, then p is connected to r.

Connected components. Maximal set of objects that are mutually connected.



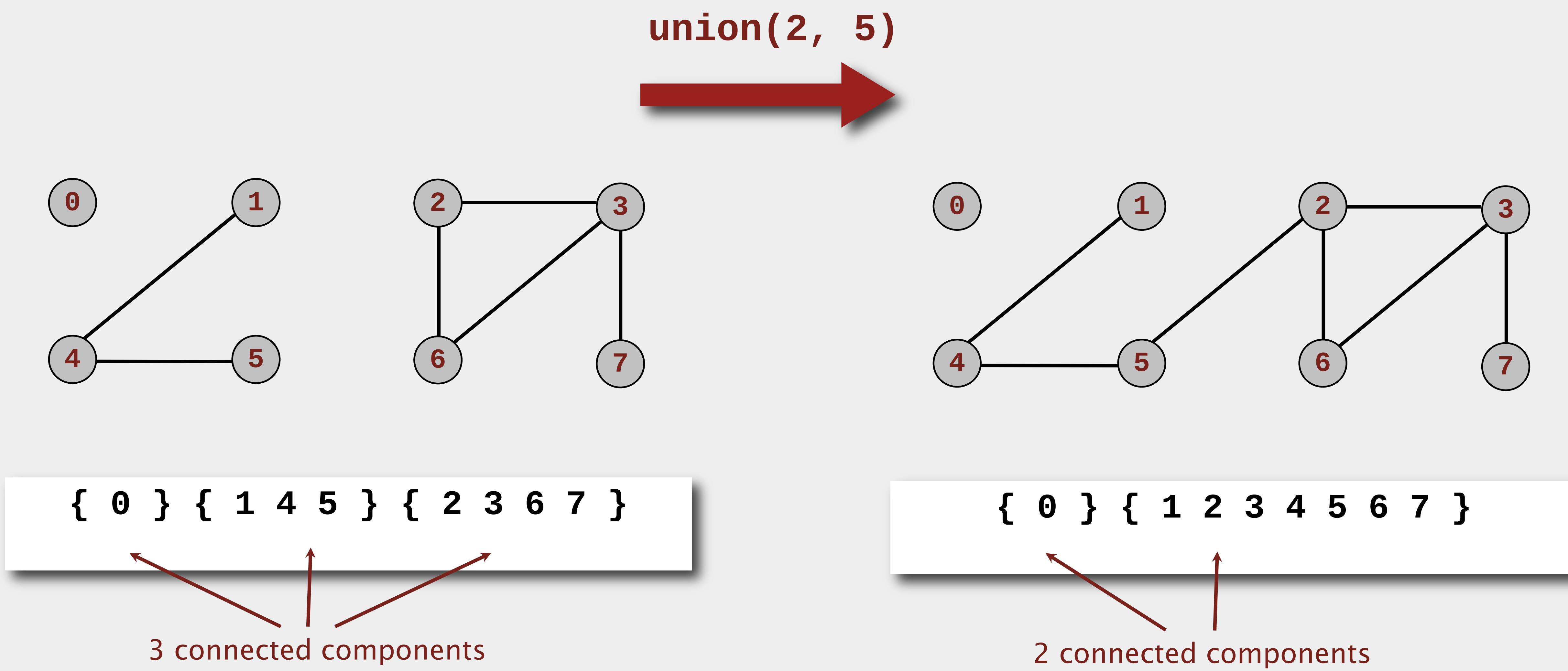
{ 0 } { 1 4 5 } { 2 3 6 7 }

3 connected components

Implementing the operations

Find query. Check if two objects are in the same component.

Union command. Replace components containing two objects with their union.



Union-find data type (API)

Goal. Design efficient data structure for union-find.

- Number of objects N can be huge.
- Number of operations M can be huge.
- Find queries and union commands may be intermixed.

public class UF	
`UF(int N)`	*initialize union-find data structure with N objects (0 to N-1)*
`void union(int p, int q)`	*add connection between p and q*
`boolean connected(int p, int q)`	*are p and q in the same component?*
`int find(int p)`	*component identifier for p (0 to N-1)*
`int count()`	*number of components*

Dynamic-connectivity client

- Read in number of objects N from standard input.
- Repeat:
  - read in pair of integers from standard input
  - write out pair if they are not already connected

```
public static void main(String[] args)
{
   int N = StdIn.readInt();
   UF uf = new UF(N);
   while (!StdIn.isEmpty())
   {
      int p = StdIn.readInt();
      int q = StdIn.readInt();
      if (uf.connected(p, q)) continue;
      uf.union(p, q);
      StdOut.println(p + " " + q);
   }
}
```

```
% more tiny.txt
10
4 3
3 8
6 5
9 4
2 1
8 9
5 0
7 2
6 1
1 0
6 7
```

- dynamic connectivity
- quick find
- quick union
- improvements
- applications

Quick-find [eager approach]

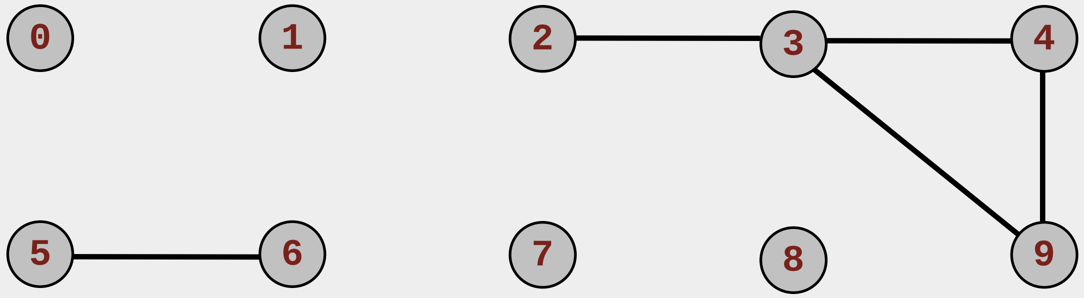
Data structure.

- Integer array **`id[]`** of size **`N`**.
- Interpretation: **`p`** and **`q`** in same component iff they have the same id.

i	0	1	2	3	4	5	6	7	8	9
id[i]	0	1	9	9	9	6	6	7	8	9

5 and 6 are connected

2, 3, 4, and 9 are connected



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Find. Check if `p` and `q` have the same id.

`id[3] = 9`; `id[6] = 6`

3 and 6 in different components

Quick-find [eager approach]

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Find. Check if `p` and `q` have the same id.

`id[3] = 9`; `id[6] = 6`

3 and 6 in different components

Union. To merge sets containing `p` and `q`, change all entries with `id[p]` to `id[q]`.

i	0	1	2	3	4	5	6	7	8	9
id[i]	0	1	6	6	6	6	6	7	8	6

union of 3 and 6

2, 3, 4, 5, 6, and 9 are connected

problem: many values can change

Quick-find example

		id[]									
p	q	0	1	2	3	4	5	6	7	8	9
4	3	0	1	2	3	4	5	6	7	8	9
		0	1	2	3	3	5	6	7	8	9
3	8	0	1	2	3	3	5	6	7	8	9
		0	1	2	8	8	5	6	7	8	9
6	5	0	1	2	8	8	5	6	7	8	9
		0	1	2	8	8	5	5	7	8	9
9	4	0	1	2	8	8	5	5	7	8	9
		0	1	2	8	8	5	5	7	8	8
2	1	0	1	2	8	8	5	5	7	8	8
		0	1	1	8	8	5	5	7	8	8
8	9	0	1	1	8	8	5	5	7	8	8
5	0	0	1	1	8	8	5	5	7	8	8
		0	1	1	8	8	0	0	7	8	8
7	2	0	1	1	8	8	0	0	7	8	8
		0	1	1	8	8	0	0	1	8	8
6	1	0	1	1	8	8	0	0	1	8	8
		1	1	1	8	8	1	1	1	8	8
1	0	1	1	1	8	8	1	1	1	8	8
6	7	1	1	1	8	8	1	1	1	8	8

id[p] and id[q] differ, so union() changes entries equal to id[p] to id[q] (in red)

id[p] and id[q] match, so no change

Quick-find: Java implementation

```
public class QuickFindUF
{
   private int[] id;

   public QuickFindUF(int N)
   {
      id = new int[N];
      for (int i = 0; i < N; i++)
         id[i] = i;
   }

   public boolean connected(int p, int q)
   {  return id[p] == id[q];  }

   public void union(int p, int q)
   {
      int pid = id[p];
      int qid = id[q];
      for (int i = 0; i < id.length; i++)
         if (id[i] == pid) id[i] = qid;
   }
}
```

set id of each object to itself
(N array accesses)

check whether **p** and **q**
are in the same component
(2 array accesses)

change all entries with **id[p]** to **id[q]**
(linear number of array accesses)

Quick-find is too slow

Cost model. Number of array accesses (for read or write).

algorithm	init	union	find
quick-find	N	N	1

Quick-find defect.

- Union too expensive.
- Trees are flat, but too expensive to keep them flat.
- Ex. Takes N^2 array accesses to process sequence of N union commands on N objects.

- dynamic connectivity
- quick find
- quick union
- improvements
- applications

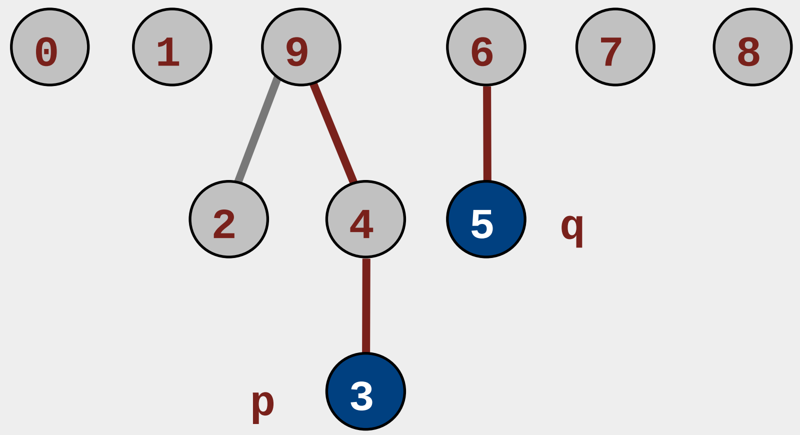
Quick-union [lazy approach]

Data structure.

- Integer array `id[]` of size `N`.
- Interpretation: `id[i]` is parent of `i`.
- Root of `i` is `id[id[id[...id[i]...]]]`.

keep going until it doesn't change

i	0	1	2	3	4	5	6	7	8	9
id[i]	0	1	9	4	9	6	6	7	8	9



3's root is 9; 5's root is 6

Quick-union [lazy approach]

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- Integer array `id[]` of size `N`.
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keep going until it doesn't change

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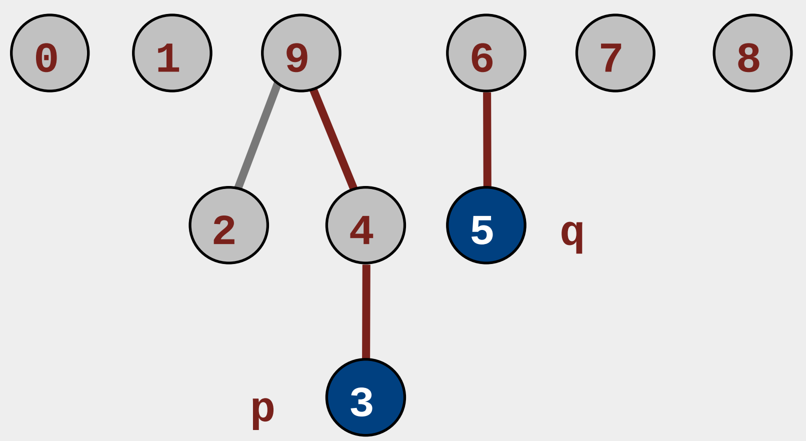
id[i]	0	1	9	4	9	6	6	7	8	9



3's root is 9; 5's root is 6

3 and 5 are in different components

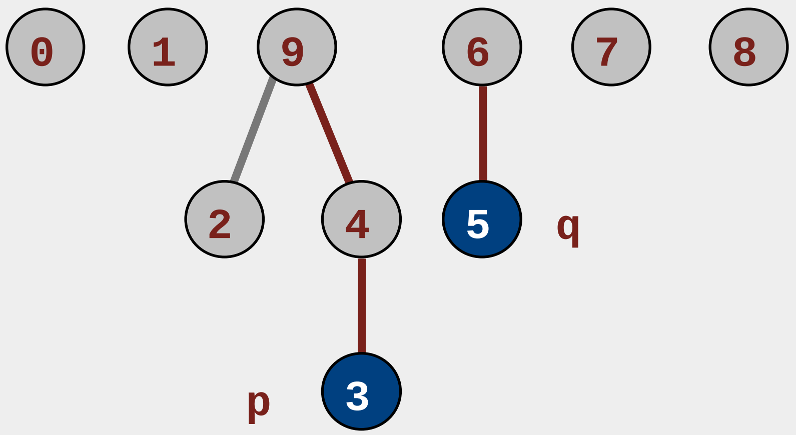
Find. Check if `p` and `q` have the same root.

Quick-union [lazy approach]

Data structure.
- Integer array `id[]` of size `N`.
- Interpretation: `id[i]` is parent of `i`.
- Root of `i` is `id[id[id[...id[i]...]]]`.

keep going until it doesn't change

i	0	1	2	3	4	5	6	7	8	9
id[i]	0	1	9	4	9	6	6	7	8	9



3's root is 9; 5's root is 6

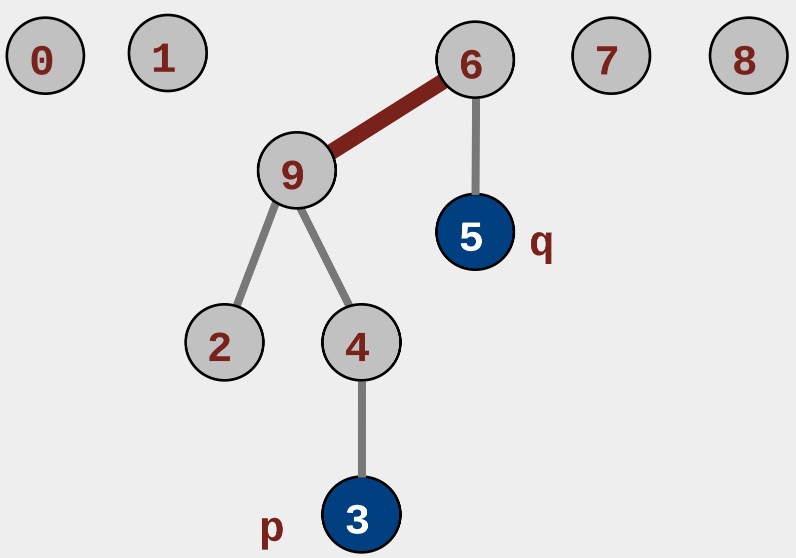
3 and 5 are in different components

Find. Check if `p` and `q` have the same root.

Union. To merge sets containing `p` and `q`, set the id of `p`'s root to the id of `q`'s root.

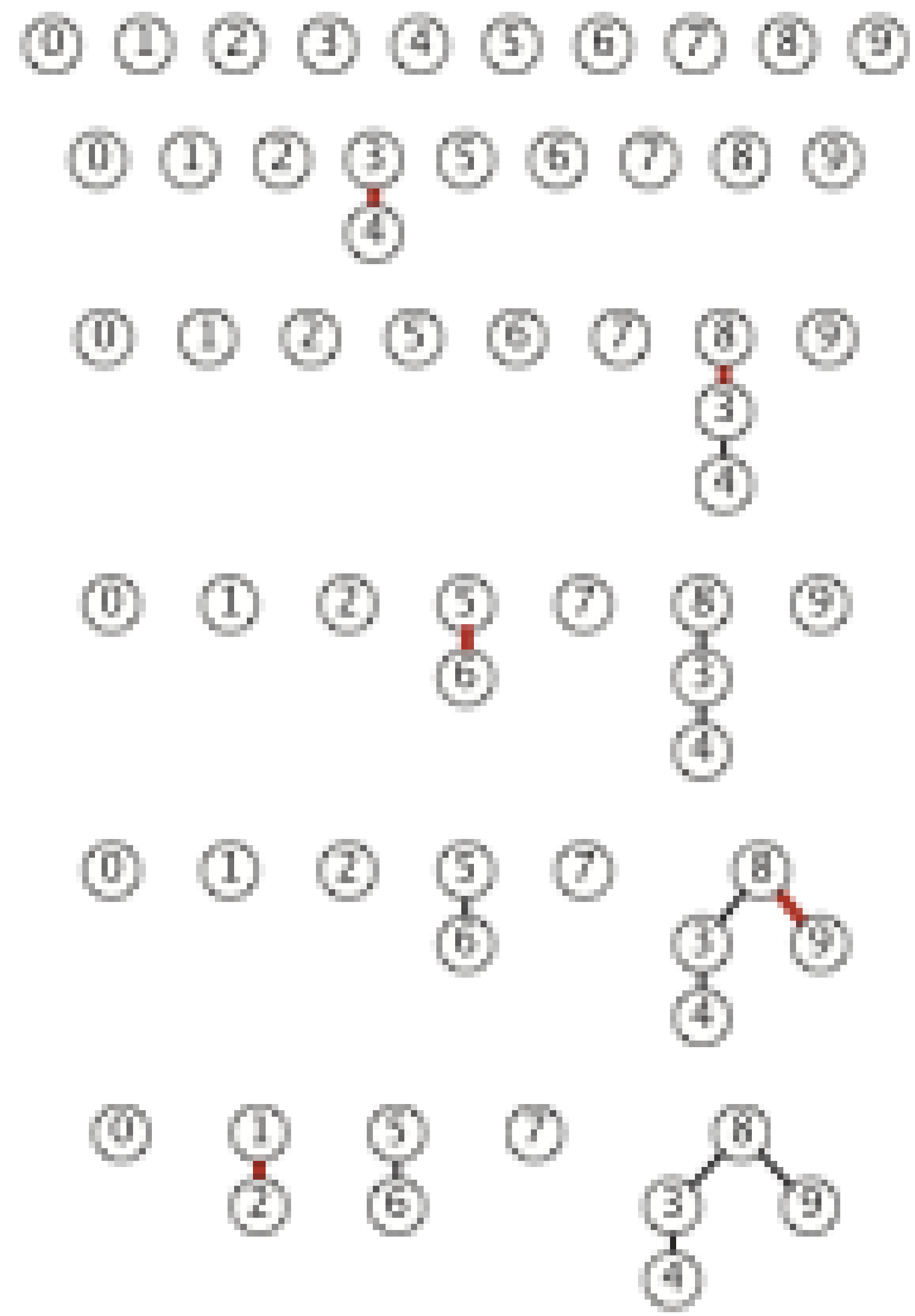
i	0	1	2	3	4	5	6	7	8	9
id[i]	0	1	9	4	9	6	6	7	8	6

only one value changes



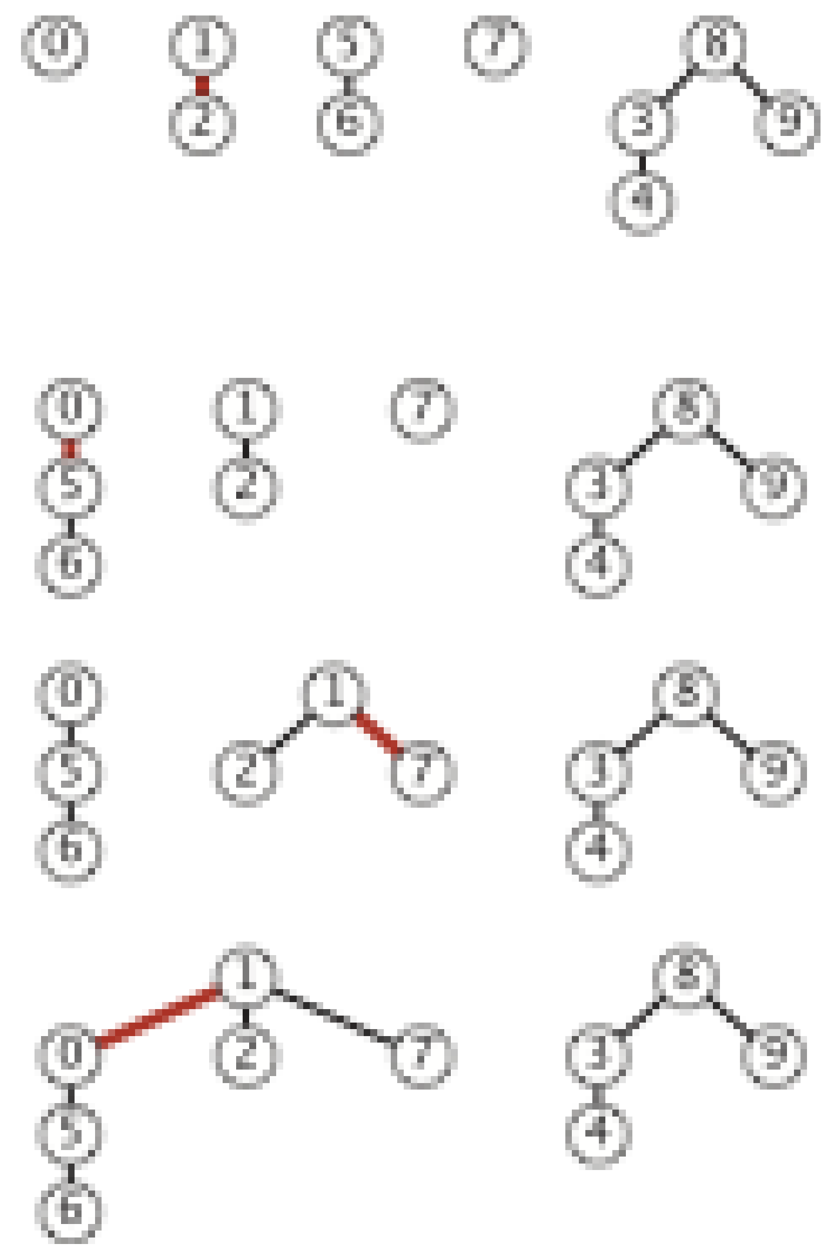
Quick-union example

p	q	id[] 0	1	2	3	4	5	6	7	8	9
4	3	0	1	2	3	4	5	6	7	8	9
		0	1	2	3	3	5	6	7	8	9
3	8	0	1	2	3	3	5	6	7	8	9
		0	1	2	8	3	5	6	7	8	9
6	5	0	1	2	8	3	5	6	7	8	9
		0	1	2	8	3	5	5	7	8	9
9	4	0	1	2	8	3	5	5	7	8	9
		0	1	2	8	3	5	5	7	8	8
2	1	0	1	2	8	3	5	5	7	8	8
		0	1	1	8	3	5	5	7	8	8



Quick-union example

p	q	0	1	2	3	4	5	6	7	8	9
8	9	0	1	1	8	3	5	5	7	8	8
5	0	0	1	1	8	3	5	5	7	8	8
		0	1	1	8	3	0	5	7	8	8
7	2	0	1	1	8	3	0	5	7	8	8
		0	1	1	8	3	0	5	1	8	8
6	1	0	1	1	8	3	0	5	1	8	8
		1	1	1	8	3	0	5	1	8	8
1	0	1	1	1	8	3	0	5	1	8	8
6	7	1	1	1	8	3	0	5	1	8	8



Quick-union: Java implementation

```
public class QuickUnionUF{   private int[] id;
public QuickUnionUF(int N)   {      id = new
int[N];
      for (int i = 0; i < N; i++) id[i] = i;   }
private int root(int i)   {
      while (i != id[i]) i = id[i];
      return i;   }   public boolean connected(int
p, int q)   {      return root(p) == root(q);   }
public void union(int p, int q)   {
      int i = root(p), j = root(q);
      id[i] = j;   }
}
```

set id of each object to itself
(N array accesses)

chase parent pointers until reach root
(depth of i array accesses)

check if p and q have same root
(depth of p and q array accesses)

change root of p to point to root of q
(depth of p and q array accesses)

Quick-union is also too slow

Cost model. Number of array accesses (for read or write).

algorithm	init	union	find
quick-find	N	N	1
quick-union	N	N †	N

← worst case

† includes cost of finding root

Quick-find defect.

- Union too expensive (N array accesses).
- Trees are flat, but too expensive to keep them flat.

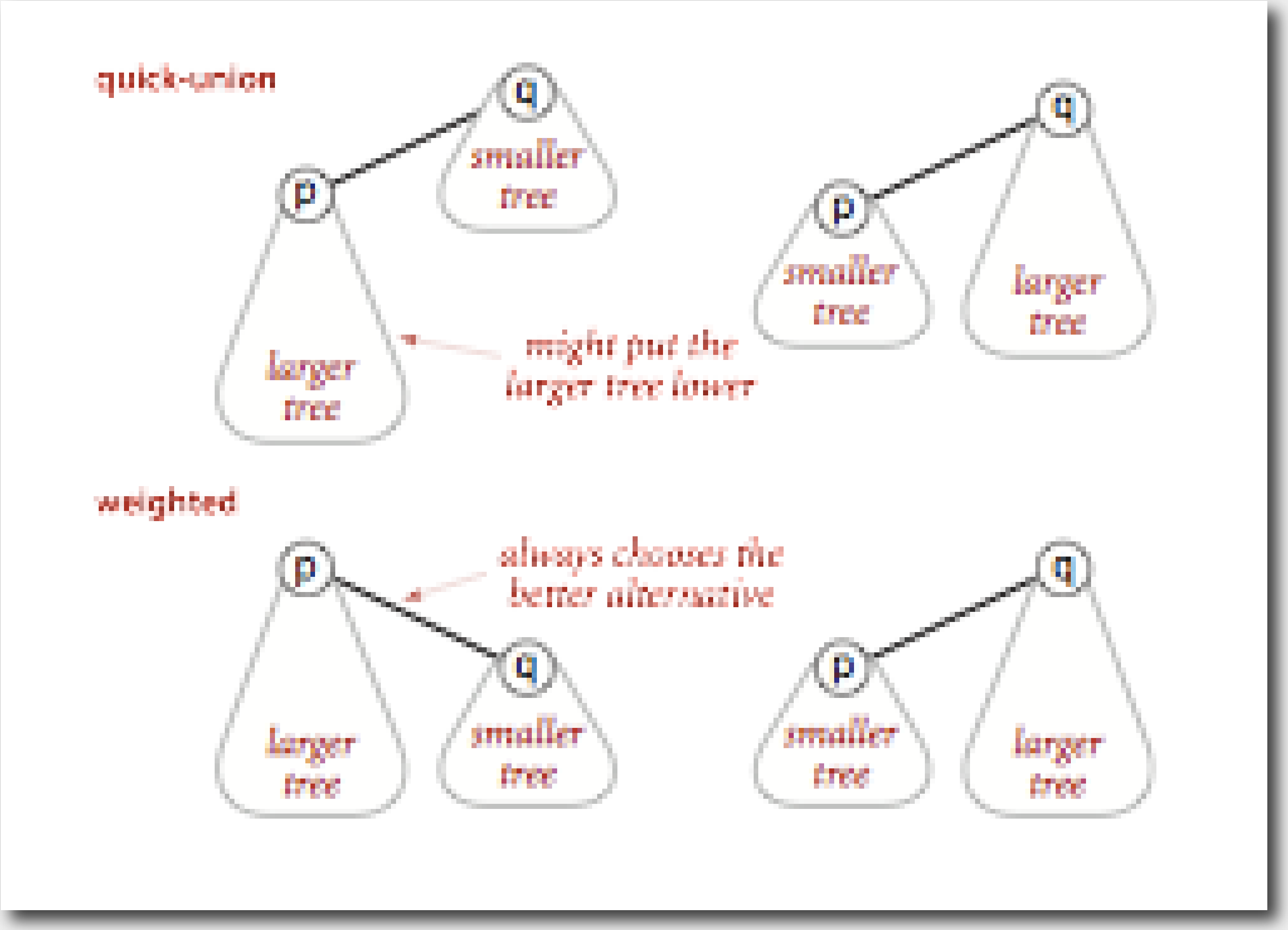
Quick-union defect.

- Trees can get tall.
- Find too expensive (could be N array accesses).

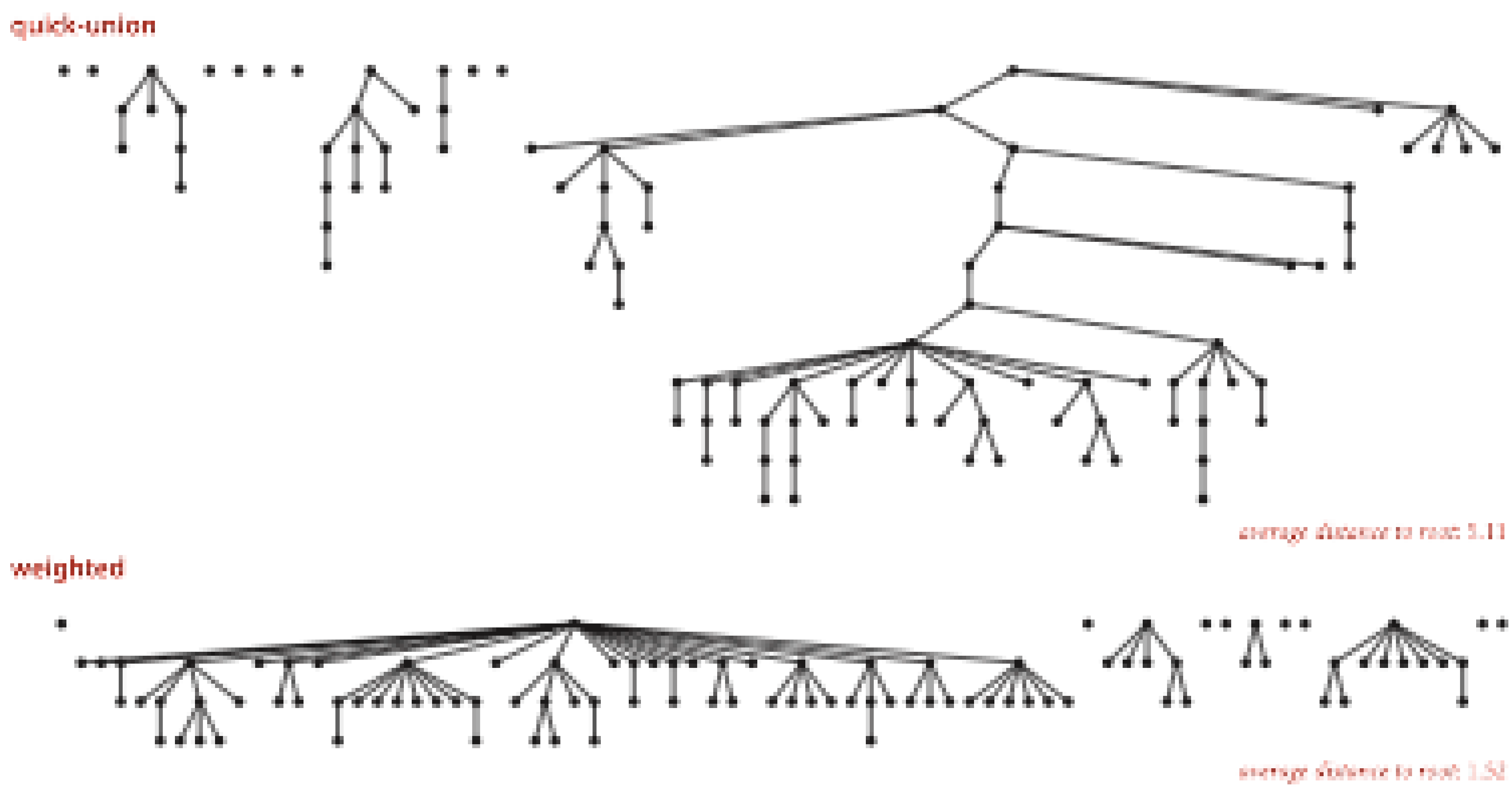
Improvement 1: weighting

Weighted quick-union.

- Modify quick-union to avoid tall trees.
- Keep track of size of each tree (number of objects).
- Balance by linking small tree below large one.



Quick-union and weighted quick-union example



Quick-union and weighted quick-union (100 sites, 88 union() operations)

Weighted quick-union: Java implementation

Data structure. Same as quick-union, but maintain extra array `sz[i]` to count number of objects in the tree rooted at `i`.

Find. Identical to quick-union.

```
return root(p) == root(q);
```

Union. Modify quick-union to:

- Merge smaller tree into larger tree.
- Update the `sz[]` array.

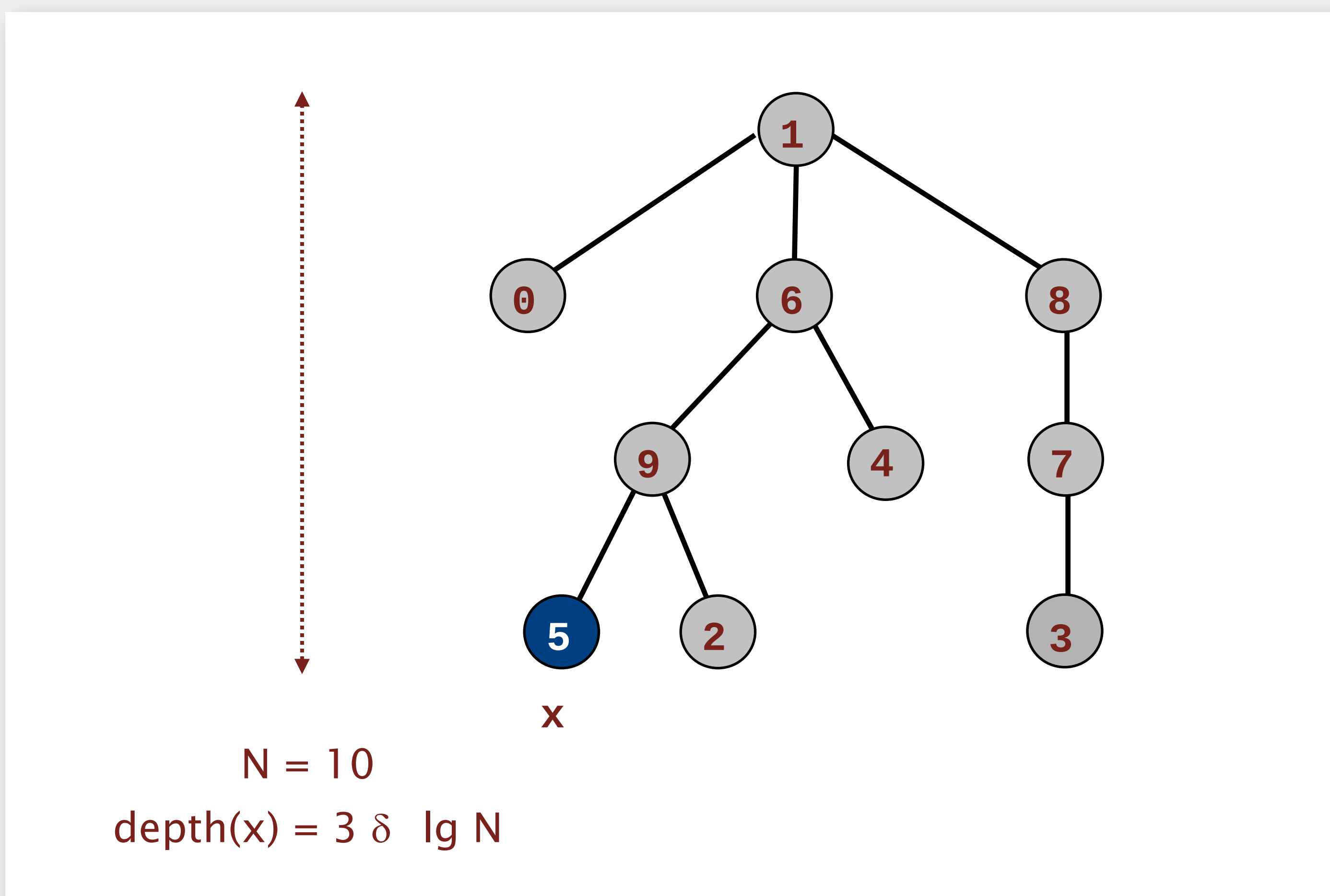
```
int i = root(p);
int j = root(q);
if  (sz[i] < sz[j]) { id[i] = j; sz[j] += sz[i]; }
else                { id[j] = i; sz[i] += sz[j]; }
```

Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p and q.
- Union: takes constant time, given roots.

Proposition. Depth of any node x is at most $\lg N$.



Weighted quick-union analysis

Running time.

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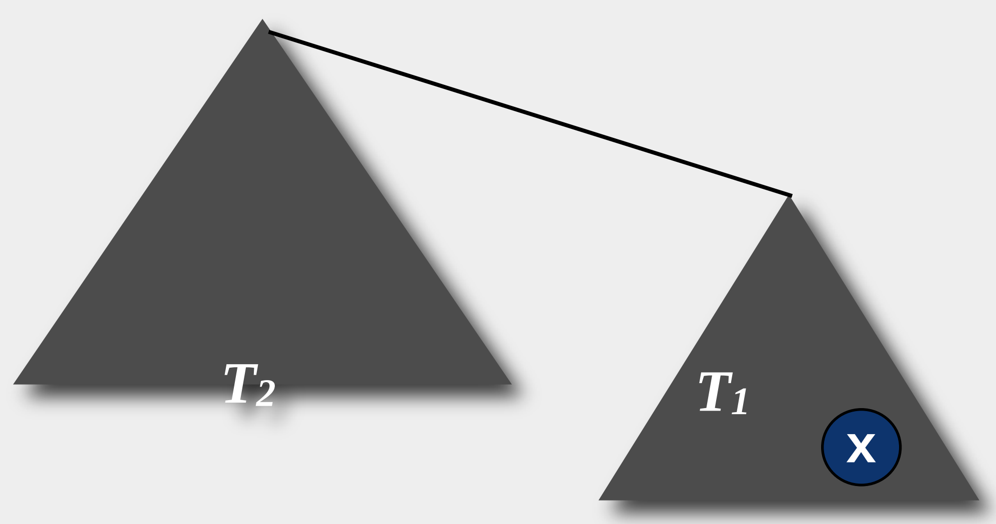
Proposition. Depth of any node x is at most $\lg N$.

Pf. When does depth of x increase?

Increases by 1 when tree T_1 containing x is merged into another tree T_2.

- The size of the tree containing x at least doubles since $| T_2 |$ ε $| T_1 |$.
- Size of tree containing x can double at most $\lg N$ times. Why?



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algorithm	init	union	find
quick-find	N	N	1
quick-union	N	N †	N
weighted QU	N	lg N †	lg N

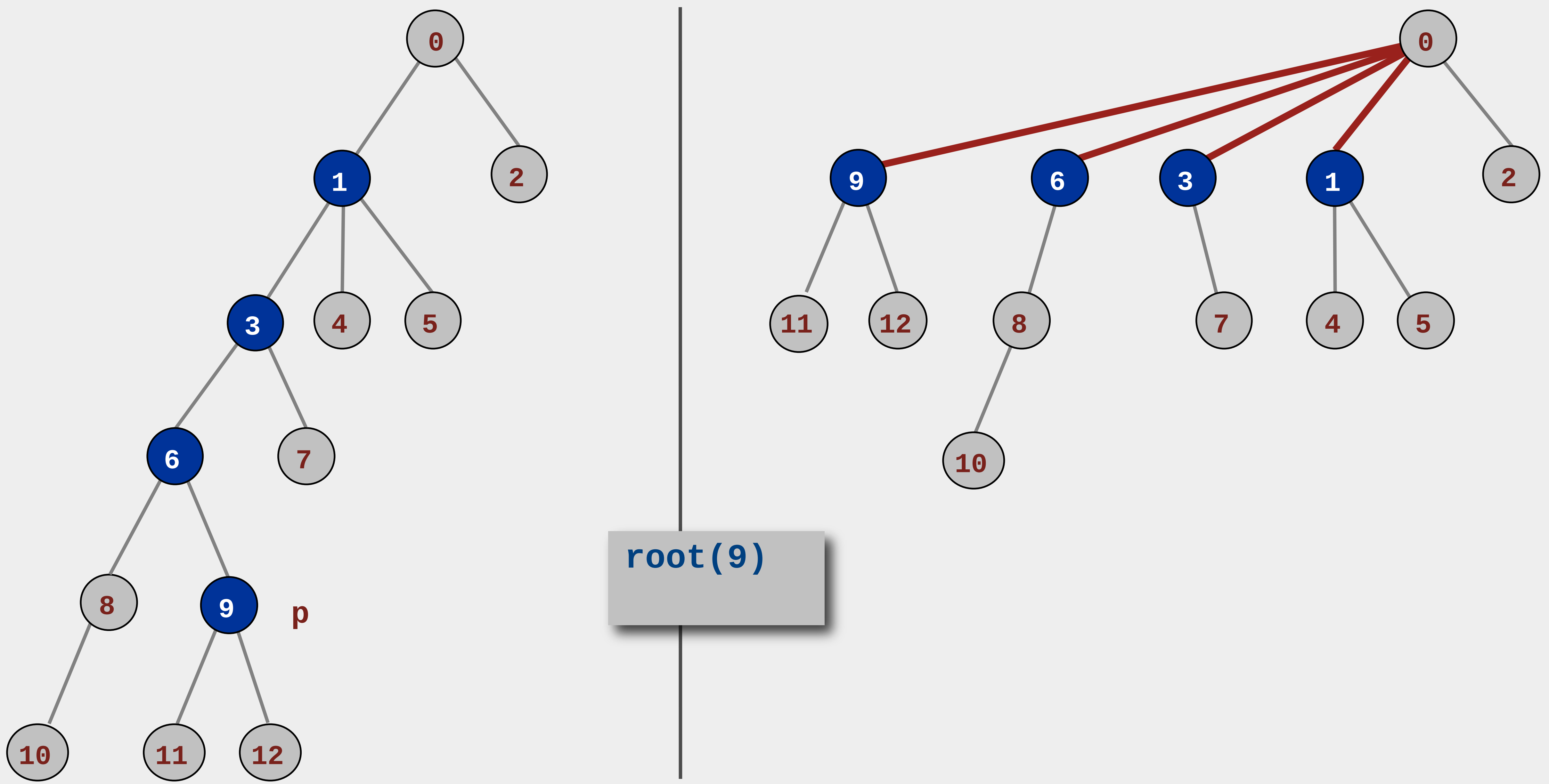
† includes cost of finding root

Q. Stop at guaranteed acceptable performance?

A. No, easy to improve further.

Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the id of each examined node to point to that root.



Path compression: Java implementation

Standard implementation: add second loop to `find()` to set the `id[]` of each examined node to the root.

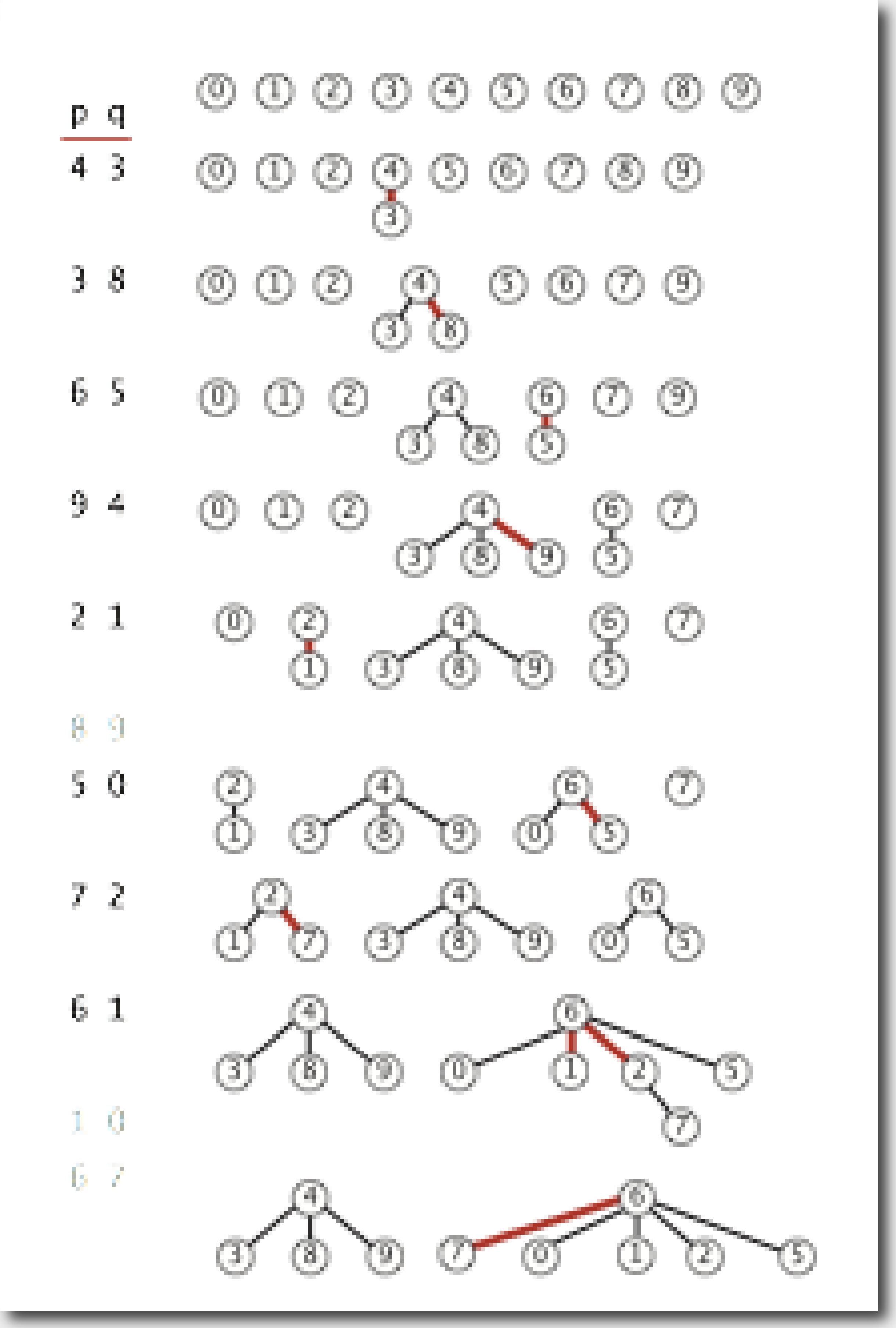
Simpler one-pass variant: halve the path length by making every other node in path point to its grandparent.

```
public int root(int i)
{
   while (i != id[i])
   {
      id[i] = id[id[i]];
      i = id[i];
   }
   return i;
}
```

only one extra line of code !

In practice. … most completely flat.

Weighted quick-union with path compression example



1 linked to 6 because of path compression

7 linked to 6 because of path compression

Weighted quick-union with path compression: amortized analysis

Bob Tarjan
(Turing Award '86)

Proposition. Starting from an empty data structure, any sequence of M union–find operations on N objects makes at most proportional to $N + M \lg^* N$ array accesses.

- Proof is very difficult.
- Can be improved to $N + M \langle (M, N)$.
- But the algorithm is still simple!

see COS 423

Linear-time algorithm for M union-find ops on N objects?

- Cost within constant factor of reading in the data.
- In theory, WQUPC is not quite linear.
- In practice, WQUPC is linear.

because $\lg^* N$ is a constant in this universe

Amazing fact. No linear-time algorithm exists.

in "cell-probe" model of computation

N	lg* N
1	0
2	1
4	2
16	3
65536	4
2^{65536}	5

lg* function

Summary

Bottom line. WQUPC makes it possible to solve problems that could not otherwise be addressed.

algorithm	worst-case time
quick-find	M N
quick-union	M N
weighted QU	N + M log N
QU + path compression	N + M log N
weighted QU + path compression	N + M lg* N

M union-find operations on a set of N objects

Ex. [10^9 unions and finds with 10^9 objects]

- WQUPC reduces time from 30 years to 6 seconds.
- Supercomputer won't help much; good algorithm enables solution.

- dynamic connectivity
- quick find
- quick union
- improvements
- applications

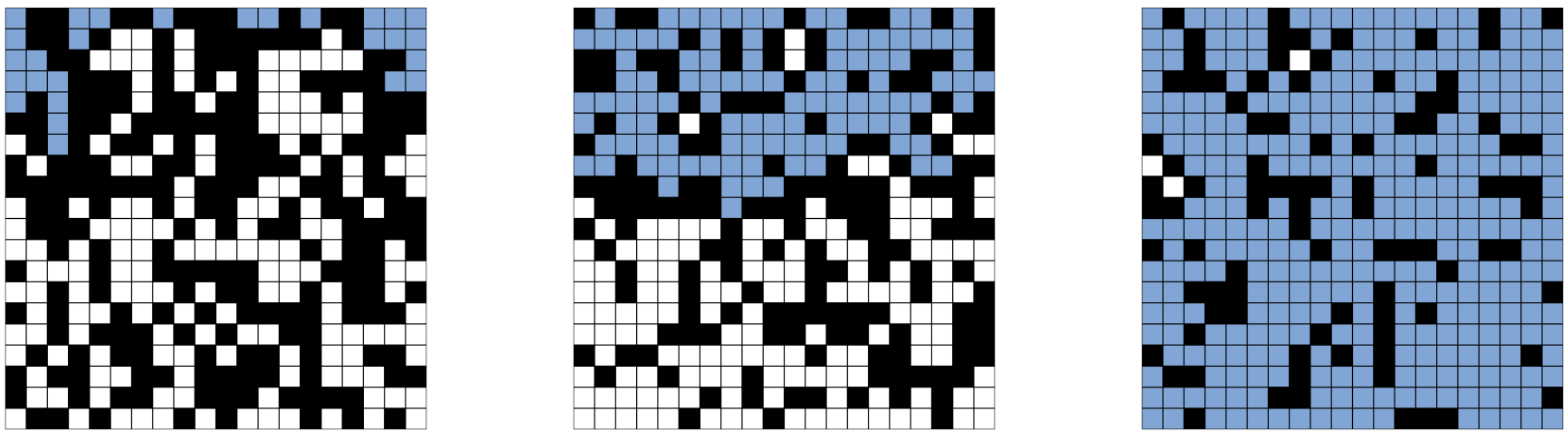
A model for many physical systems:

- N-by-N grid of sites.
- Each site is open with probability p (or blocked with probability $1 - p$).
- System percolates iff top and bottom are connected by open sites.

model	system	vacant site	occupied site	percolates
electricity	material	conductor	insulated	conducts
fluid flow	material	empty	blocked	porous
social interaction	population	person	empty	communicates

Likelihood of percolation

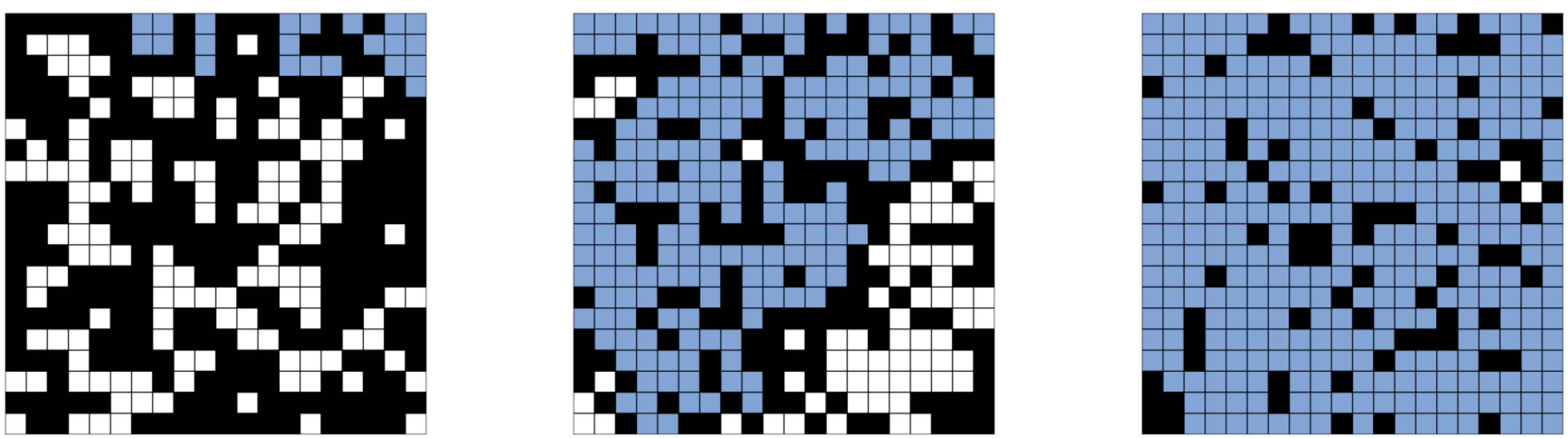
Depends on site vacancy probability p.



p low (0.4)
does not percolate

p medium (0.6)
percolates?

p high (0.8)
percolates

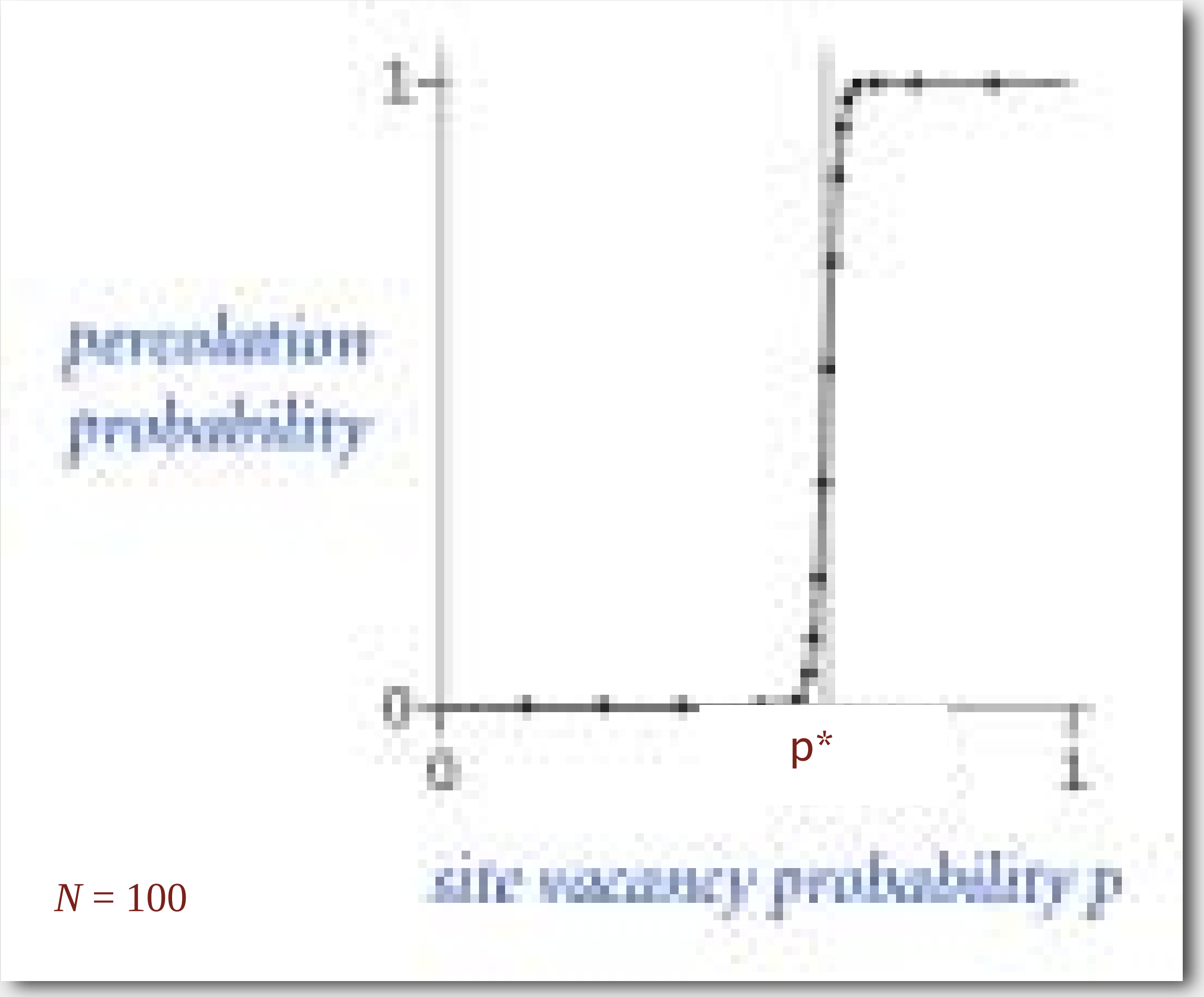


Percolation phase transition

When N is large, theory guarantees a sharp threshold p^*.

- $p > p^*$: almost certainly percolates.
- $p < p^*$: almost certainly does not percolate.

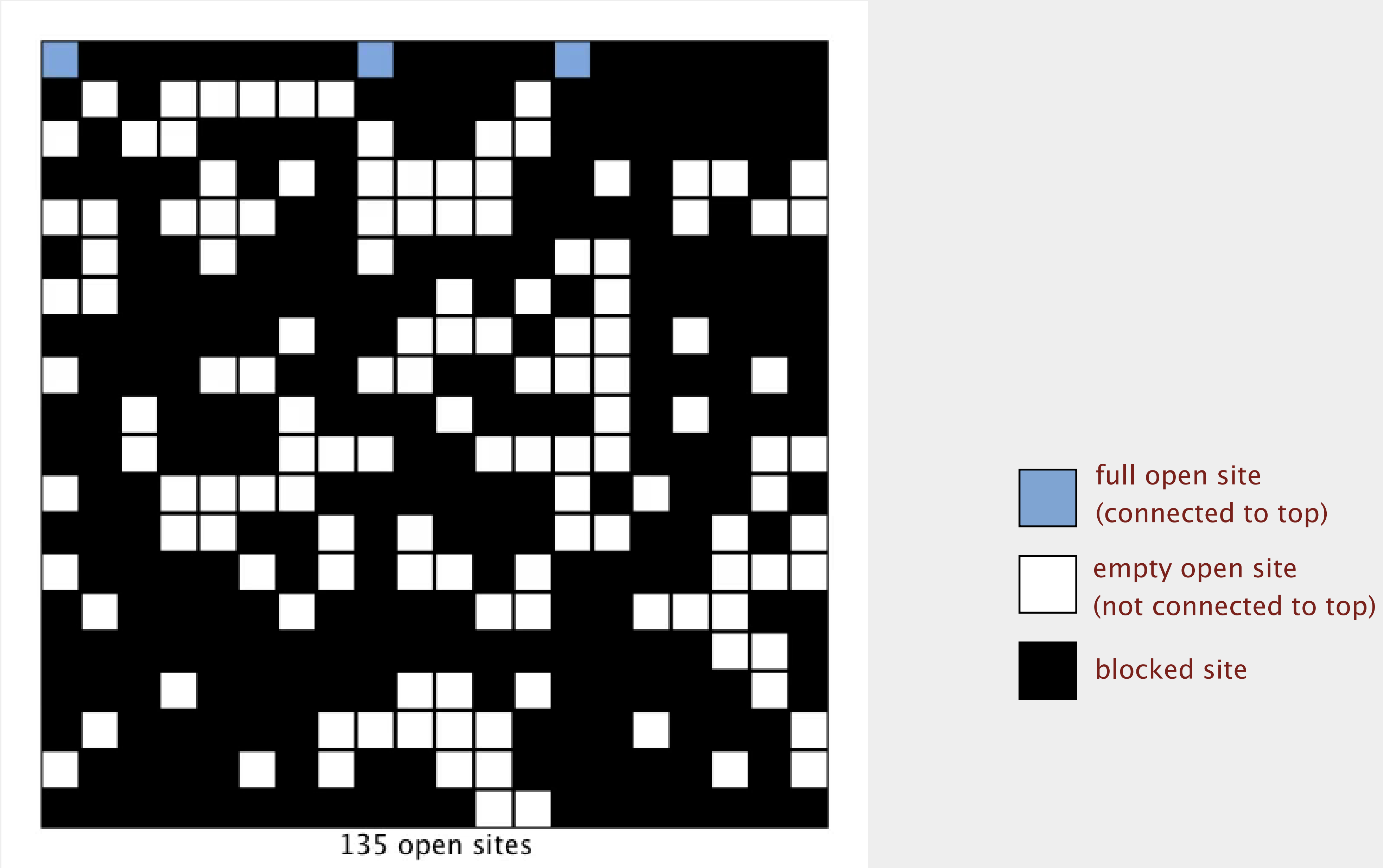
Q. What is the value of p^* ?



Monte Carlo simulation

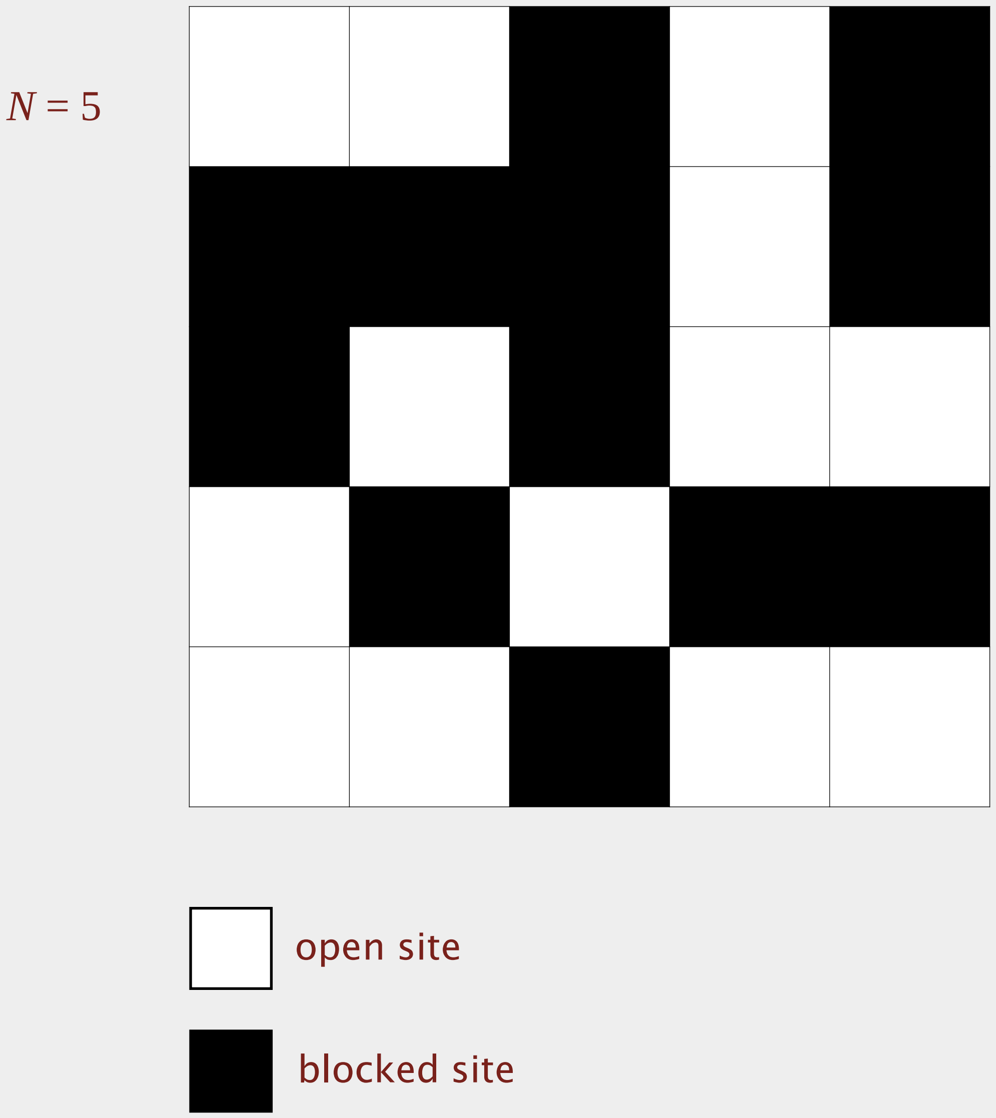
- Initialize N-by-N whole grid to be blocked.
- Declare random sites open until top connected to bottom.
- Vacancy percentage estimates p^*.

$N = 20$



Dynamic connectivity solution to estimate percolation threshold

Q. How to check whether an N-by-N system percolates?

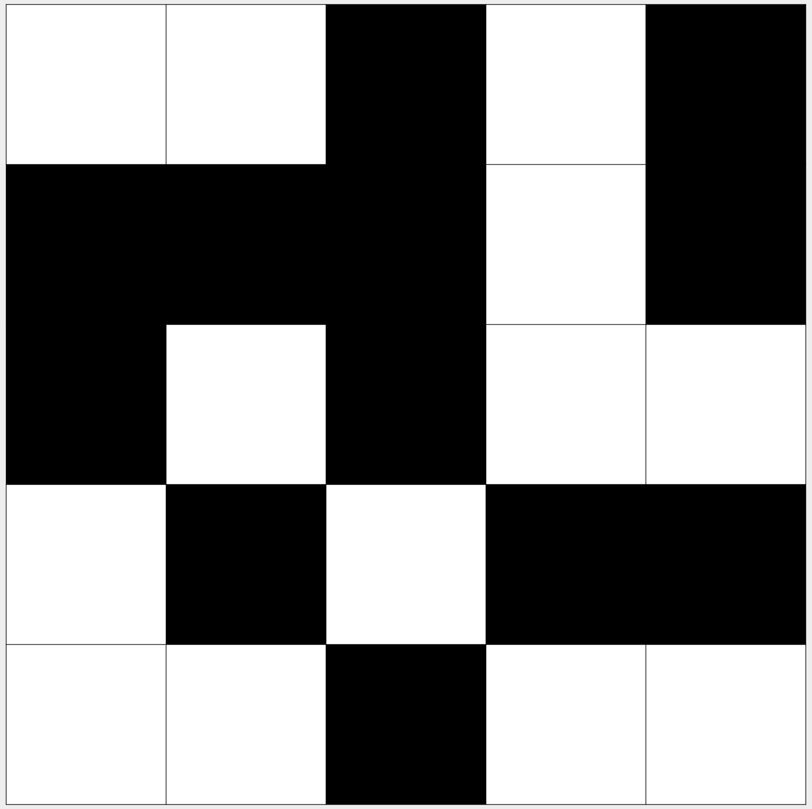


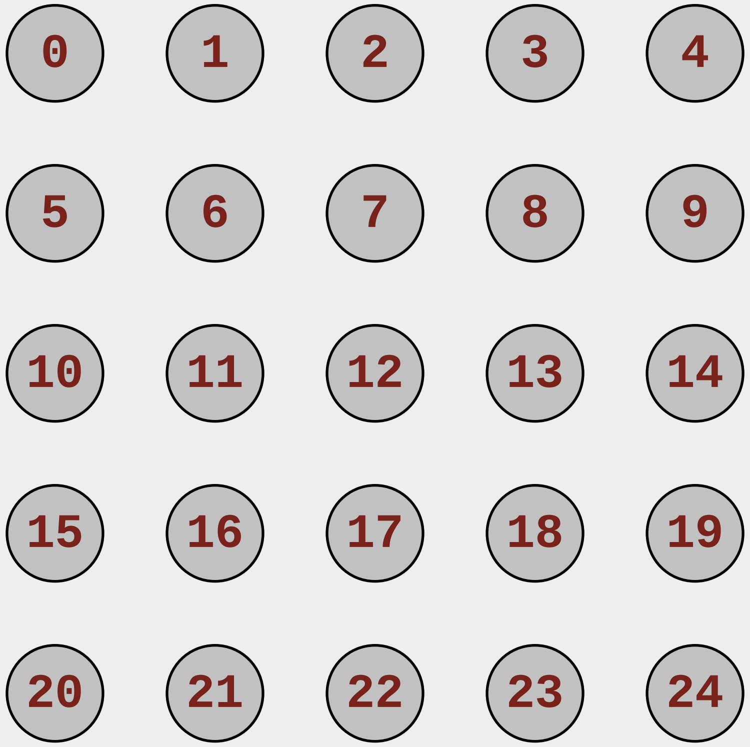
Dynamic connectivity solution to estimate percolation threshold

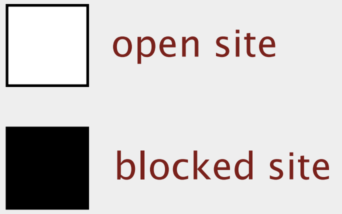
Q. How to check whether an N-by-N system percolates?

- Create an object for each site and name them 0 to $N^2 - 1$.

$N = 5$

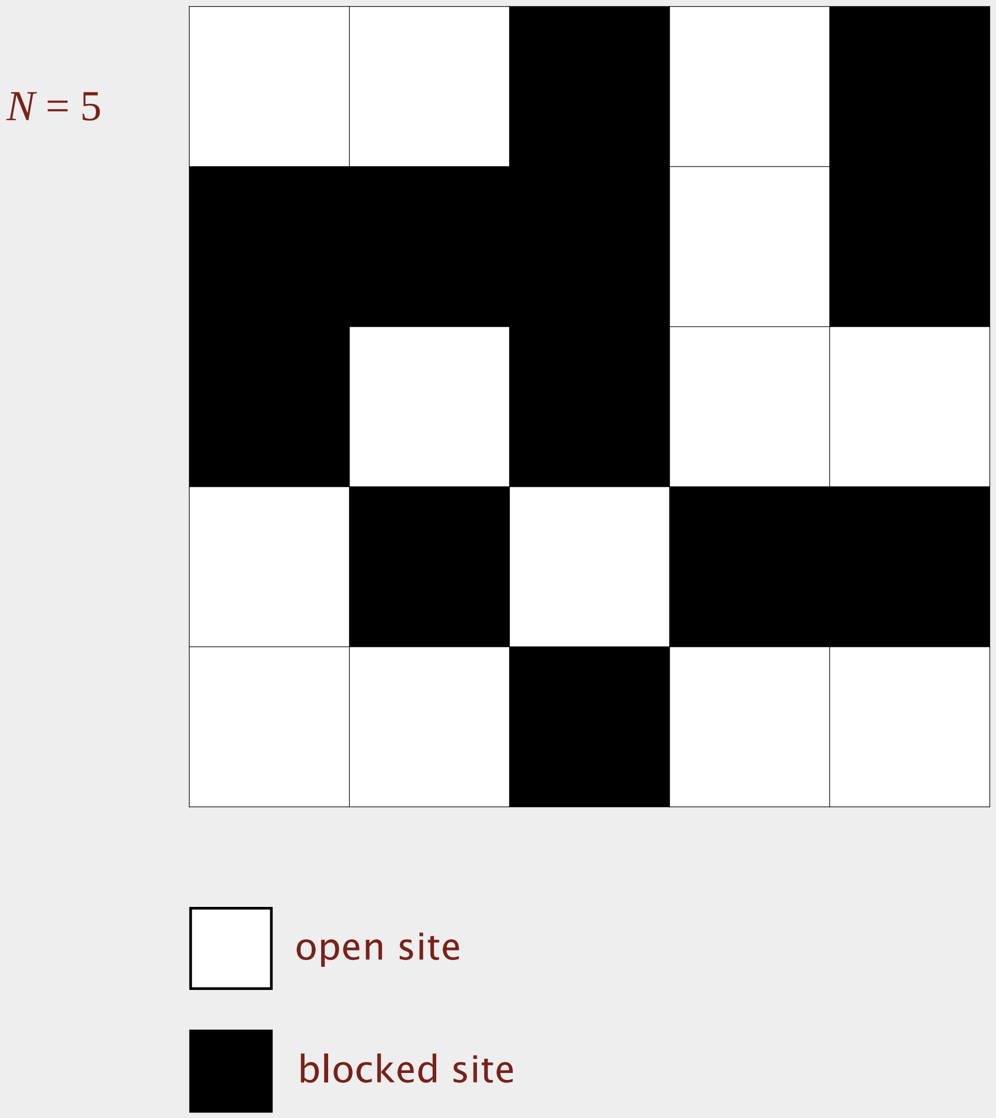


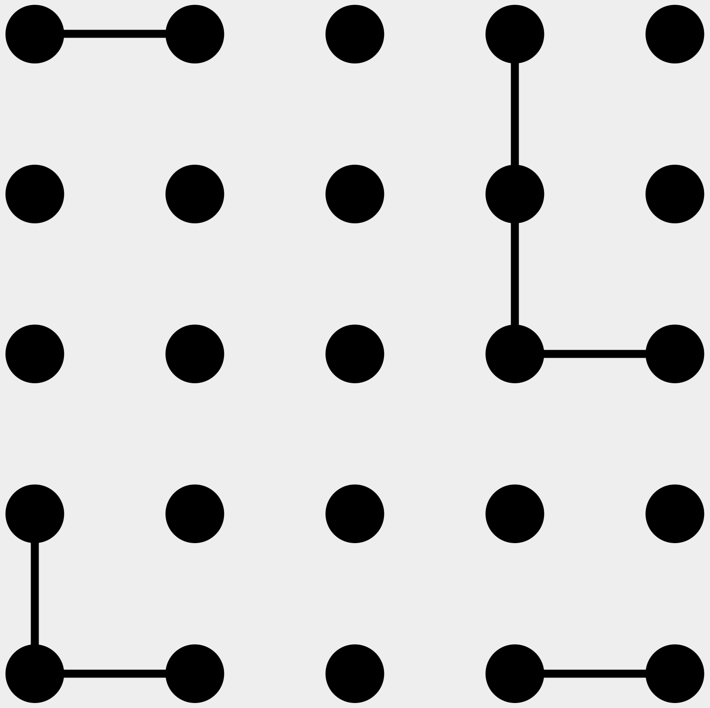




Q. How to check whether an N-by-N system percolates?

- Create an object for each site and name them 0 to $N^2 - 1$.
- Sites are in same set if connected by open sites.





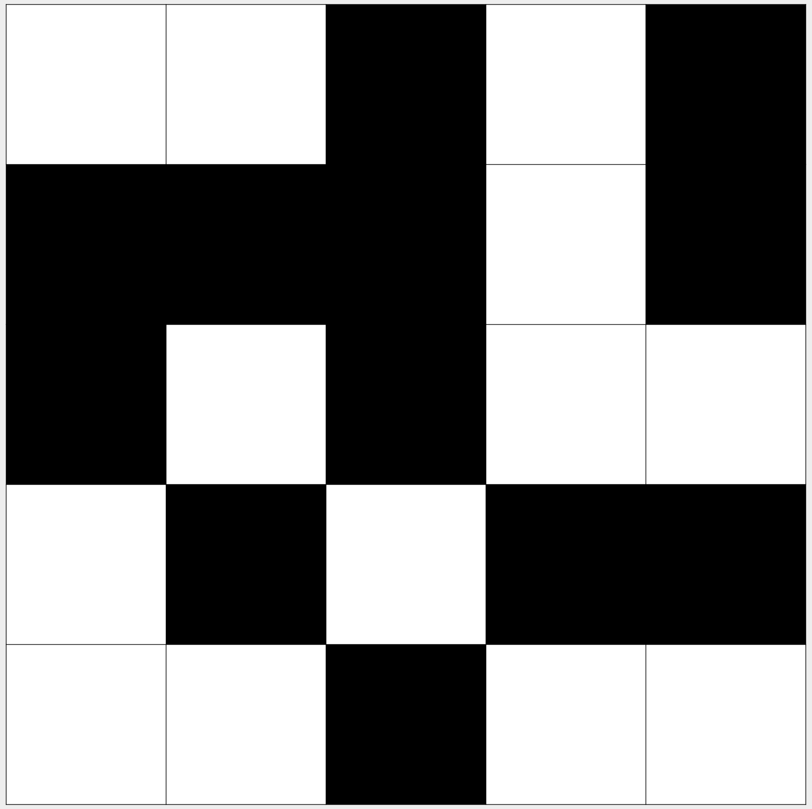
Dynamic connectivity solution to estimate percolation threshold

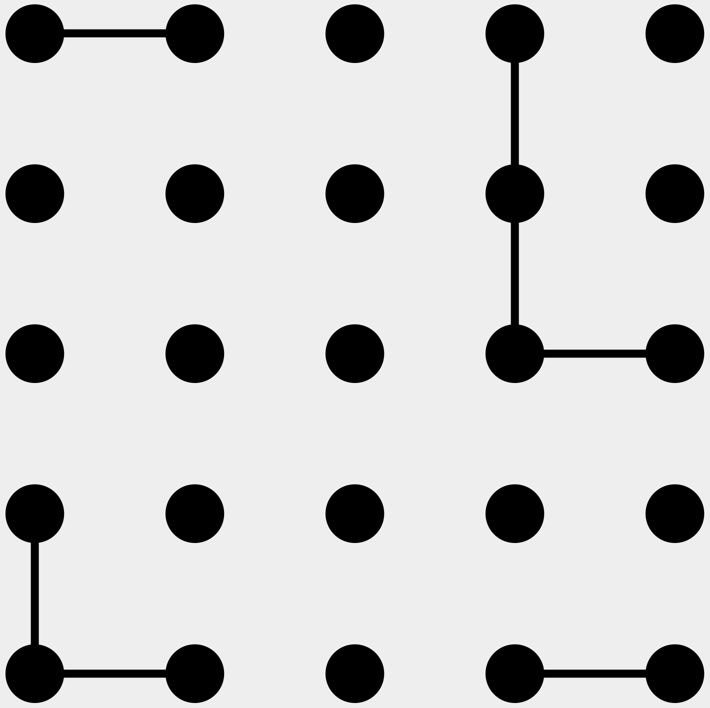
Q. How to check whether an N-by-N system percolates?

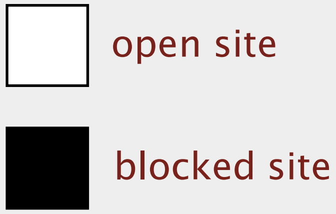
- Create an object for each site and name them 0 to $N^2 - 1$.
- Sites are in same set if connected by open sites.
- Percolates iff any site on bottom row is connected to site on top row.

brute-force algorithm: N^2 calls to connected()

$N = 5$







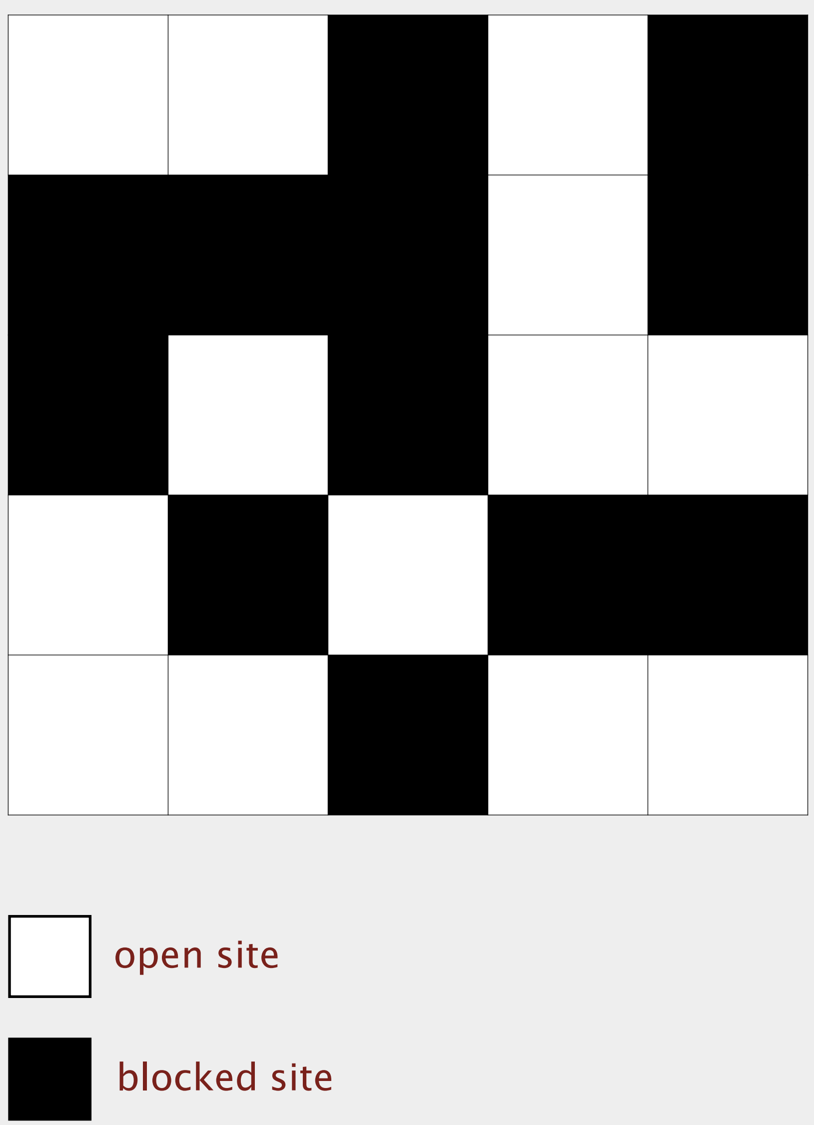
Dynamic connectivity solution to estimate percolation threshold

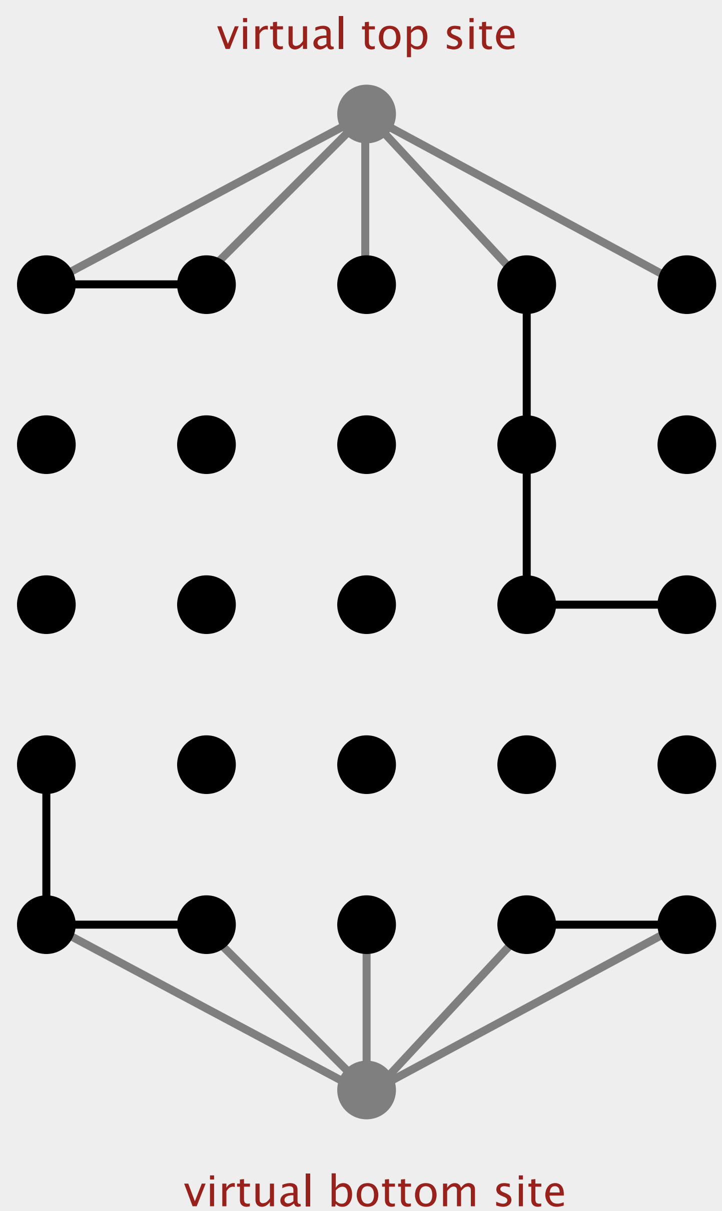
Clever trick. Introduce two virtual sites (and connections to top and bottom).

- Percolates iff virtual top site is connected to virtual bottom site.

efficient algorithm: only 1 call to connected()

$N = 5$





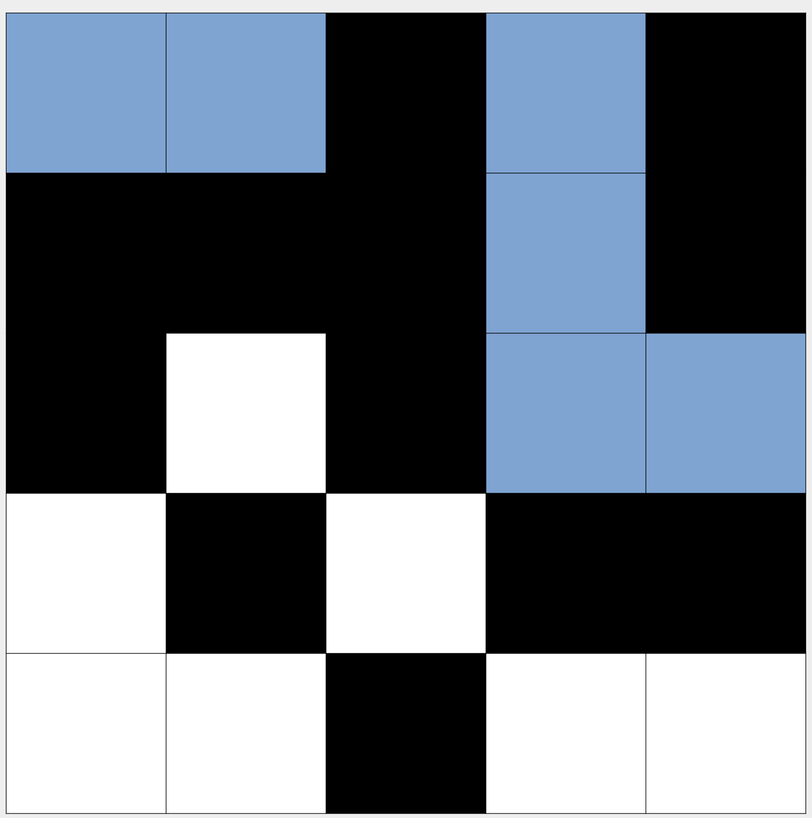
Dynamic connectivity solution to estimate percolation threshold

Clever trick. Introduce two virtual sites (and connections to top and bottom).

- Percolates iff virtual top site is connected to virtual bottom site.
- Open site is full iff connected to virtual top site.

needed only for visualization

$N = 5$

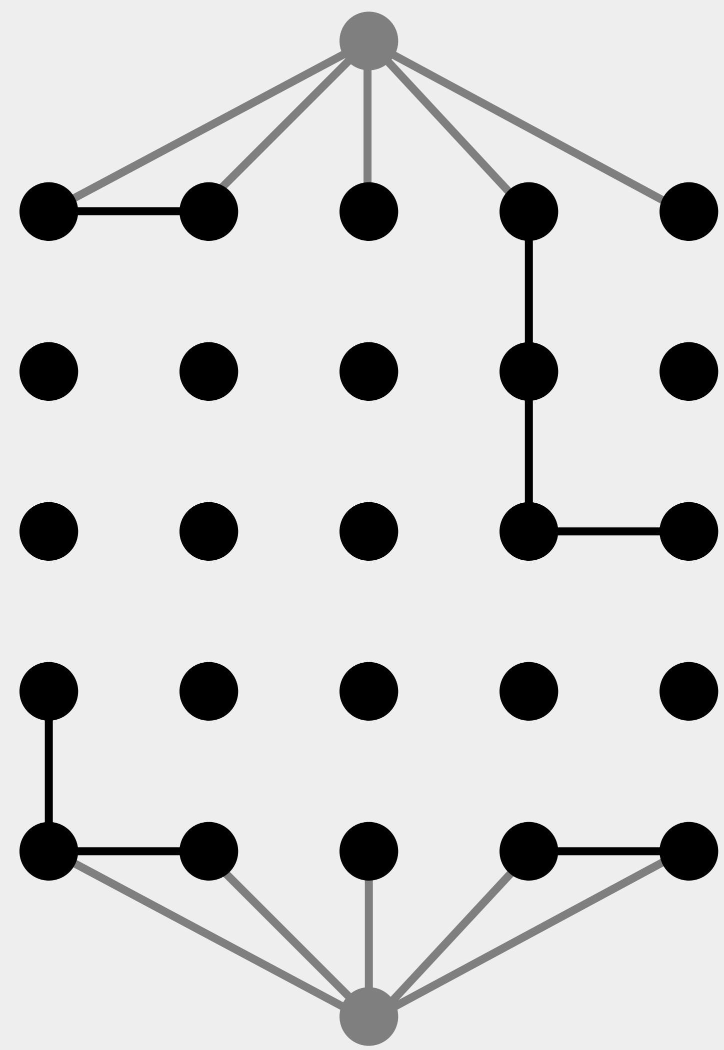


empty open site (not connected to top)

full open site (connected to top)

blocked site

virtual top site

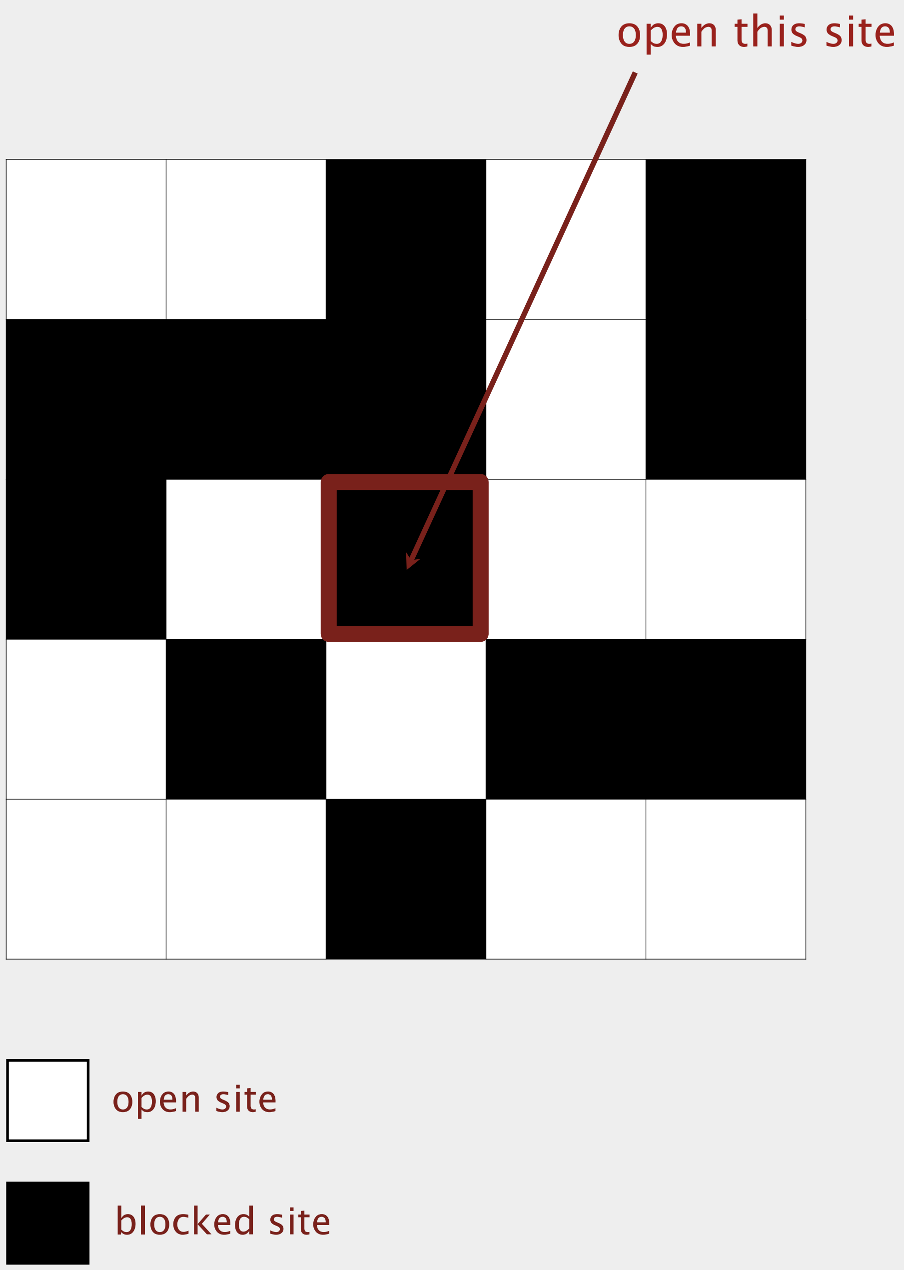


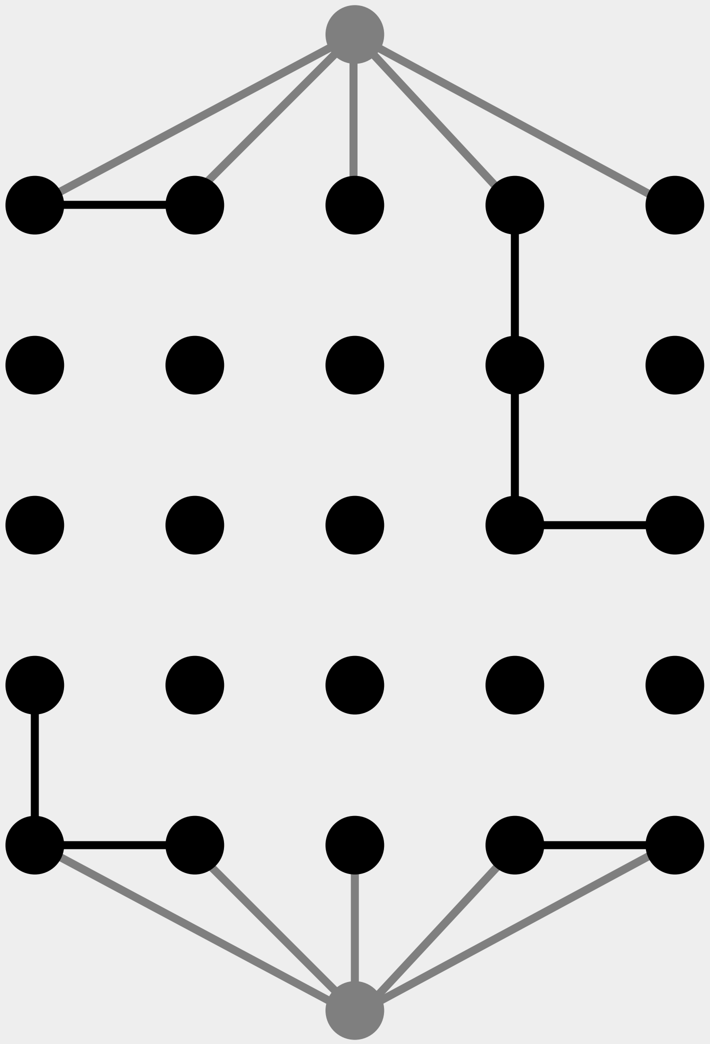
virtual bottom site

Dynamic connectivity solution to estimate percolation threshold

Q. How to model as dynamic connectivity problem when opening a new site?

$N = 5$





Dynamic connectivity solution to estimate percolation threshold

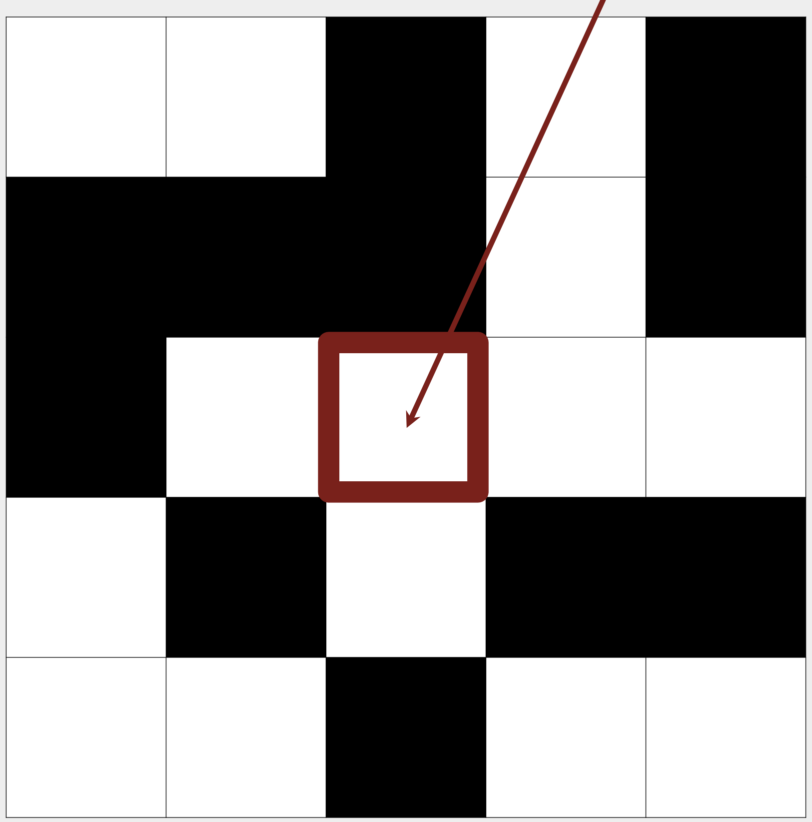
Q. How to model as dynamic connectivity problem when opening a new site?

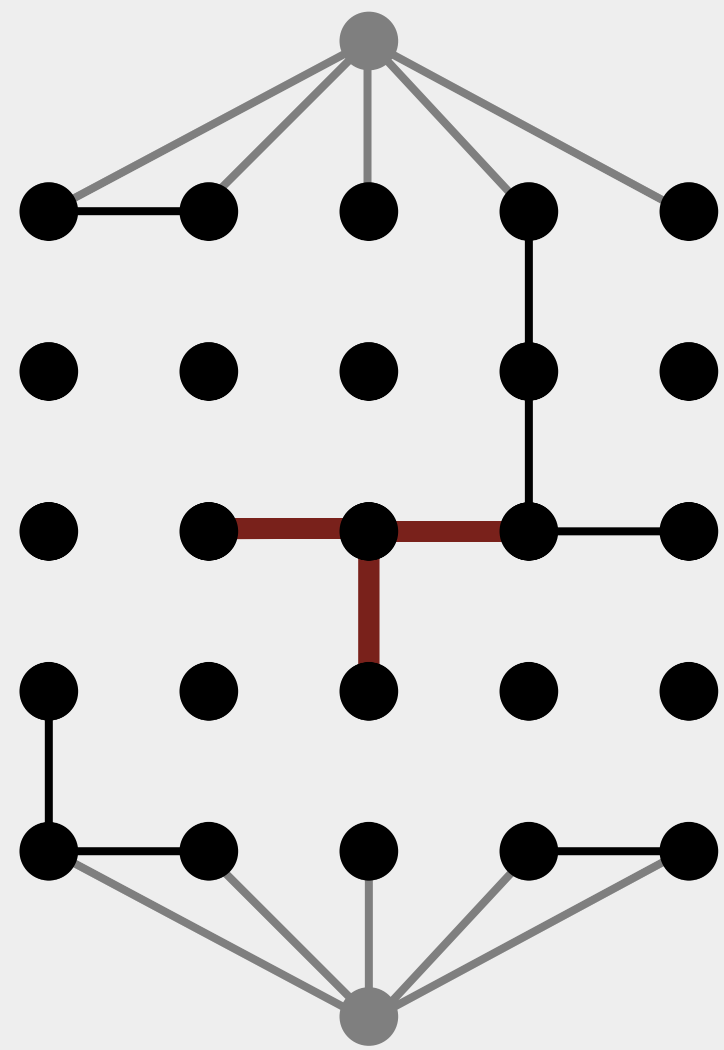
A. Connect new site to all of its adjacent open sites.

up to 4 calls to union()

open this site

$N = 5$





open site

blocked site

Subtext of today's lecture (and this course)

Steps to developing a usable algorithm.

- Model the problem.
- Find an algorithm to solve it.
- Fast enough? Fits in memory?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.

The scientific method.

Mathematical analysis.