1. (20 points) A deck of 10 cards, each bearing a distinct number from 1 to 10, is shuffled to mix the cards thoroughly. Three cards are removed one at a time from the deck. What is the probability that the three cards are selected in an increasing order?

2. (10 points) Show that $E(c) = c$. In other words, the random variable maps any outcome of the sample space to constant $c$.

3. (10 points) Given a random variable $X$, how does the variance of $cX$ related to that of $X$, where $c$ is a constant.

4. (10 points) Which is larger: the expectation of the square of a random variable, or the square of its expectation? Explain your choice.

5. (10 points) Show that for any random variable $X$ that takes on only the values of 0 and 1, we have $V(X) = E(X)(1 - E(X))$.

   *Hint: Start with the definition of $V(X)$ and use algebraic manipulations in the previous three problems.*

6. (20 points) A carnival game consists of three dice in a cage. A player can bet a dollar on any of the numbers 1 through 6. The cage is shaken, and the payoff is as follows. If the player’s number doesn’t appear on any of the dice, he loses his dollar. Otherwise, if his number appears on exactly $k$ of the three dice, for $k = 1, 2, 3$, he keeps the dollar and wins $k$ more dollars. What is his expected gain from playing the carnival game once?

7. (20 points) Consider the process of randomly tossing identical balls into 5 bins, numbered 1, 2, ..., 5. The tosses are independent, and on each toss the ball is equally likely to end up in any bin.

   Let us call a toss in which a ball falls into an empty bin a “hit”. What is the average number $n$ of tosses required to get 5 hits.

   *Hint: The hits can be used to partition the tosses into stages. The $i$th stage consists of the tosses after the $(i - 1)$st hit until the $i$th hit. The first stage consists of the first toss, since we are guaranteed to have a hit when all bins are empty. For each toss during the $i$th stage, figure out what is the probability to obtain a hit. Let $n_i$ denotes the number of tosses in the $i$th stage. What kind of distribution does $n_i$ have? What is its expectation? Read textbook P. 433. Example 10.*