A set is an unordered collection of objects. We write \( a \in A \) to denote that \( a \) is an element of the set \( A \). The notation \( a \notin A \) denotes that \( a \) is not an element of the set \( A \).

There are several ways to describe a set:

1. List all the members of a set.
2. Use set builder notation.
3. Use Venn diagram to represent graphically.

Special sets:

1. **Empty set** is a set that has no elements, denoted by \( \emptyset \).
2. **Singleton set** is a set that has one element, e.g., \( \{\emptyset\} \).

The set \( A \) is said to be a subset of \( B \) if and only if every element of \( A \) is also an element of \( B \), denoted by \( A \subseteq B \). For every set \( S \), \( \emptyset \subseteq S \), and \( S \subseteq S \). \( A \) is a **proper subset** of \( B \) if \( A \) is a subset of \( B \) but that \( A \neq B \).

The **cardinality** of a set \( S \), denoted by \(|S|\), represents the number of distinct elements in \( S \). If \( S \) is a finite set with \( n \) distinct elements, then \(|S| = n\). What happens if \( S \) has infinitely many distinct elements? Now consider two sets \( A \) and \( B \), where \( A \subseteq B \). If \( B \) is a finite set, we conclude that \(|A| < |B|\). What happens if \( B \) is a infinite set? What can you say about general sets \( A \) and \( B \) if \(|A| \leq |B|\)?

The **power set** of a set \( S \), denoted by \( P(S) \), is the set of all subsets of the set \( S \). For example, given \( S = \{0, 1, 2\} \), the power set \( P(S) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\} \). What relationship can you find between \(|S|\) and \(|P(S)|\)?

We define the following **set operations**:

1. union: \( A \cup B \)
2. intersection: \( A \cap B \)
3. difference: \( A - B \)
4. complement: \( \overline{A} \).

Note: \(|A \cup B| = |A| + |B| - |A \cap B|\), which is called **inclusion-exclusion principle**. Show it using Venn diagram.

A few set identities that be established using Venn diagram.

\[
\begin{align*}
(1) & \quad A \cup \emptyset = A, \quad A \cap U = A \\
(2) & \quad A \cup U = U, \quad A \cap \emptyset = \emptyset \\
(3) & \quad A \cup A = A, \quad A \cap A = A \\
(4) & \quad \overline{\overline{A}} = A \\
(5) & \quad A \cup B = B \cup A, \quad A \cap B = B \cap A \\
(6) & \quad A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C \\
(7) & \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\
\end{align*}
\]
\[ (8) \quad \overline{A \cup B} = \overline{A} \cap \overline{B}, \quad \overline{A \cap B} = \overline{A} \cup \overline{B} \]
\[ (9) \quad A \cup (A \cap B) = A, \quad A \cap (A \cup B) = A \]
\[ (10) \quad A \cup \overline{A} = U, \quad A \cap \overline{A} = \emptyset \]

Next, we will answer all the questions left for the infinite sets. We start with the discussion about functions.

Let \( A \) and \( B \) be nonempty sets. A function from \( A \) to \( B \) is an assignment of exactly one element of \( B \) to each element of \( A \). Some examples are

1. One-to-one function: if \( f(a) = f(b) \), then \( a = b \).
2. Onto function: \( \forall b \in B, \exists a \in A, \text{such that } f(a) = b \).
3. Bijection function: one-to-one & onto

If \( f : A \to B \) is a bijection, then

1. Inverse function, denoted by \( f^{-1} \).
2. \( |A| = |B| \).

Definition: A set that is either finite or has the same cardinality as the set of positive integers is called countable. A set that is not countable is called uncountable. If an infinite set \( S \) is countable, we denote its cardinality by \( \aleph_0 \). Intuitively speaking, an infinite set is countable if and only if it is possible to list the elements of the set in a sequence indexed by the positive integers.

We consider a few examples:

1. Set of odd positive integers is countable
2. Set of all integers is countable
3. Set of positive rational numbers is countable.

We can now see that for infinite sets \( A \) and \( B \), where \( A \subset B \), \( |A| \) might equal to \( |B| \).

Question: does uncountable set exist?