Lecture 22, Matrix

Matrix is an rectangular array of numbers. An $m \times n$ matrix has m rows and n columns. If $m = n$, we call it a square matrix. Two matrices are equal if they have the same number of rows and columns, and the corresponding entries in every position are equal.

We usually represent a matrix as follows:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & 0 \end{bmatrix}$$

Matrix arithmetic: Let A and B be both $m \times n$ matrices,

(1) $A + B = [a_{ij} + b_{ij}]$
(2) $kA = [ka_{ij}]$, k constant

The product of matrices A and B is defined if A is $m \times k$ and B is $k \times n$ matrices. In other words, AB is defined if the column number of A is the same as the row number of B. The dimensionality of AB is $m \times n$. The entries in AB are defined by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}.$$

The task is to take the ith row of A, and the jth column of B, multiply the corresponding element, and sum the result. For square matrices, we can then define the power of the matrices.

Example 1.

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

Notice matrix product is not commutative in general.

Example 2. *Let*

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

Then,

$$AB = \begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix}, \quad BA = \begin{bmatrix} 4 & 3 \\ 3 & 3 \end{bmatrix}$$

There is a special square matrix called identity matrix:

$$I_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Entries are all zero other than the ones on the diagonal.

Given any $m \times n$ matrix A, we have

$$AI_n = I_m A = A$$

There is an counterpart of the division operation in matrix operations (requiring square matrices). It is called the inverse of matrix, denoted by A^{-1}. An $n \times n$ matrix A has an inverse if there exists a matrix B such that

$$AB = BA = I_n.$$

B is the inverse matrix of A. A matrix having inverse is called a nonsingular matrix, otherwise it is called singular. Not every matrix has inverse.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

The transpose of an $m \times n$ matrix A is an $n \times m$ matrix B, such that $a_{ij} = b_{ji}$. In other words, the ith row, jth column entry of A appears at jth row, and ith column of B. The transpose of A is usually denoted by A' or A^T. A square matrix A is called symmetric if $A = A^T$. For example, the adjacency matrix of an undirected graph is symmetric.

Path is a squence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph. The length of the path equals to the number of edges traveled.

Theorem 1. *Let G be a graph with adjacency matrix A with respect to the ordering $v_1, v_2, \ldots, v_n$. The number of different paths of length r from v_i to v_j, where r is a postive integer, equals to the (i, j)th entry of A^r.*

Proof. We use mathematical induction.

First, the number of paths from v_i to v_j of length 1 is the (i, j) entry of A, because this entry is the number of edges from v_i to v_j.

Assume that the (i, j)th entry of A^r is the number of different paths of length r from v_i to v_j. This is the induction hypothesis. Because $A^{r+1} = A^r A$, the (i, j)th entry of A^{r+1} equals

$$b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{in}a_{nj},$$

where b_{ik} is the (i, k)th entry of A^r. By induction hypothesis, b_{ik} is the number of paths of length r from v_i to v_k.

A path of length $r+1$ from v_i to v_j is made up of a path of length r from v_i to some intermediate vertex v_k and an edge from v_k to v_j. By the product rule, the number of such paths is the product of the number of paths of length r from v_i to v_k, namely, b_{ik}, and the number of edges from v_k to v_j, namely, a_{kj}. When these products are added for all possible intermediate vertices v_k, the desired result follows by the sum rule for counting. □

A^r can be used to determine the length of the shortest path between two vertices and can also be used to determine if two vertices are connected in the graph.