Lecture 24, Euler and Hamilton Paths

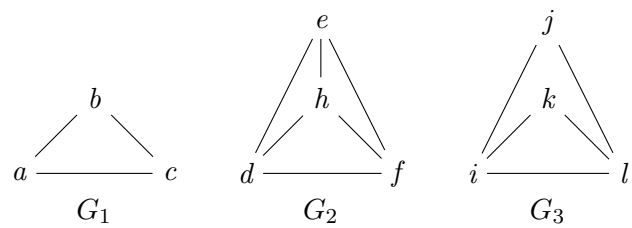
Definition 1. *An Euler circuit in a graph G is a simple circuit containing every edge of G. An Euler path in G is a simple path containing every edge of G.*

Definition 2. *A simple path in a graph G that passes through every vertex exactly once is called a Hamilton path, and a simple circuit in a graph G that passes through every vertex exactly once is called a Hamilton circuit.*

In this lecture, we will introduce a necessary and sufficient condition for the existence of Euler circuit (path). We also introduce a few sufficient conditions for the existence of Hamilton circuit. *What is the difference between sufficient and necessary?*

We start with the Euler circuit (path).

Example 1. *Consider the following three graphs.*



Only G_1 has an Euler circuit. Only G_3 has an Euler path. G_2 has neither. What is the difference?

Observations: (1) Each vertex in G_1 has even degree, and there are odd degree vertices in G_2 and G_3. (2) There are exactly 2 vertices of odd degree in G_3.

Are those observations sufficient and necessary?

Theorem 1. *A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.*

A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree

Proof. Necessary condition for the Euler circuit.

We pick an arbitrary starting vertex of the circuit, say a. The edge in the path leaving a contributes 1 to a's degree.

Every time the circuit passes through a vertex, it contributes two to the vertex's degree, one for entering and the other for leaving the vertex.

When the circuit ends, it stops at a, contributes 1 more to a's degree. Hence, every vertex will have even degree.

We show the result for the Euler path next before discussing the sufficient condition for Euler circuit.

First, suppose that a connected multigraph does have an Euler path from a to b, but not an Euler circuit. The first edge of the path contributes one to the degree of a. A contribution of two the degree of a every time the path

passes through a. The last edge in the path contributes one to the degree of b. Every time the path goes through b, it contributes two to its degree. Consequently, both a and b have odd degree. Every other vertex has even degree, because the path contributes two to the degree of a vertex whenever it passes through it.

Secondly, consider that a graph has exactly two vertices of odd degree, say a and b. If we introduce a new edge $\{a, b\}$ to G. Then there exists an Euler circuit. The removal of the new edge produces an Euler path in the original graph.

Finally, we proof the sufficient condition for the Euler circuit, which provides an approach to finding one.

We begins with an arbitrary vertex a of G. We choose an arbitrary edge incident with a and continue by building a simple path, adding edges to the path until we cannot add another edge to the path. This happens when we reach a vertex for which we have already included all edges incident with that vertex in the path.

The path we have constructed must terminate because the graph has a finite number of edges. Another claim is that the path must terminate at vertex a. To see this, every time the path goes through a vertex with even degree, it uses one edge to enter it and another to leave it. This means, other than a every time we enter a vertex, we can leave it.

An Euler circuit has been constructed if all the edges have been used. Otherwise, consider the subgraph H obtained from G be deleting the edges already used and vertices that are not incident with any remaining edges.

Because G is connected, H has at least one vertex in common with the circuit that has been deleted. Let w be such a vertex.

Every vertex in H has even degree. Beginning at w, construct a simple path in H by choosing edges as long as possible, as done in G. This path terminate at w. Next, form a circuit in G by splicing the circuit in H with the original circuit in G.

Continue the process until all edges have been used. $\square$