1. (20 points) A deck of 10 cards, each bearing a distinct number from 1 to 10, is shuffled to mix the cards thoroughly. Three cards are removed one at a time from the deck. What is the probability that the three cards are selected in an increasing order?

Solution: Each time you take three cards from the deck in a sequential order corresponds to (1) select three numbers out of 10 and (2) arrange a particular order of the numbers to be selected.

For each three number selected, there is 6 possible ways to arrange them. There is only 1 way to arrange them in an increasing order.

Therefore, we have $|\Omega| = P(n, m)$ and the event $|E| = P(n, m)/6$. The probability is 1/6.

2. (10 points) Show that $E(c) = c$. In other words, the random variable maps any outcome of the sample space to constant c.

Solution:

$$E(c) = \sum_{s \in S} p(s)X(s) = \sum_{s \in S} p(s)c = c \sum_{s \in S} p(s) = c.$$

3. (10 points) Given a random variable X, how does the variance of cX related to that of X, where c is a constant.

Solution:

$$V(cX) = E[(cX)^2] - [E(cX)]^2 = E(c^2X^2) - [cE(X)]^2 = c^2E(X^2) - c^2E(X)^2 = c^2V(X)$$

4. (10 points) Which is larger: the expectation of the square of a random variable, or the square of its expectation? Explain your choice.

Solution: $V(X) = E(X^2) - E(X)^2$, and

$$V(X) = \sum_{s \in S} [X(s) - E(X)]^2 p(s) \geq 0,$$

since each term in the summation is nonnegative, we know $E(X^2) \geq E(X)^2$.

5. (10 points) Show that for any random variable X that takes on only the values of 0 and 1, we have $V(X) = E(X)(1 - E(X))$.

   Hint: Start with the definition of $V(X)$ and use algebraic manipulations in the previous three problems.

> **Solution:** As the random variable takes either 1 or 0, we have
>
> $$E(X) = \sum_{s \in S} p(s)X(s) = \sum_{s \in S} p(s)X^2(s) = E(X^2).$$
>
> Then, we have
>
> $$V(X) = E(X^2) - E(X)^2 = E(X) - E(X)^2 = E(X)(1 - E(X))$$

6. (20 points) A carnival game consists of three dice in a cage. A player can bet a dollar on any of the numbers 1 through 6. The cage is shaken, and the payoff is as follows. If the player's number doesn't appear on any of the dice, he loses his dollar. Otherwise, if his number appears on exactly k of the three dice, for $k = 1, 2, 3$, he keeps the dollar and wins k more dollars. What is his expected gain from playing the carnival game once?

> **Solution:** Let X be the random variable that equals the times that the selected number appears on the dice. For each die, the chance that his number appears is $1/6$. The expected gain is
>
> $$-1 \cdot p(X = 0) + 1 \cdot p(X = 1) + 2 \cdot p(X = 2) + 3 \cdot p(X = 3) = -17/216.$$

7. (20 points) Consider the process of randomly tossing identical balls into 5 bins, numbered $1, 2, \ldots, 5$. The tosses are independent, and on each toss the ball is equally like to end up in any bin.

   Let us call a toss in which a ball falls into an empty bin a "hit". What is the average number n of tosses required to get 5 hits.

   Hint: The hits can be used to partition the tosses into stages. The ith stage consists of the tosses after the $(i-1)$st hit until the ith hit. The first stage consists of the first toss, since we are guaranteed to have a hit when all bins are empty. For each toss during the ith stage, figure out what is the probability to obtain a hit. Let n_i denotes the number of tosses in the ith stage. What kind of distribution does n_i have? What is its expectation?

   Read textbook P. 433. Example 10.

Solution: For each toss during the ith stage, there are $i-1$ bins that contain balls and $6-i$ empty bins. Thus, for all tosses in the ith stage, the probability of obtaining a hit is $(6-i)/5$.

n_i has a geometric distribution with probability of success $(6-i)/5$, and therefore $E(n_i) = 5/(6-i)$.

By linearity of the expectation, we have the expected number of tosses is given by

$$E(n_1 + n_2 + n_3 + n_4 + n_5) = \sum_{i=1}^{5} \frac{5}{6-i} = 5(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}) = \frac{137}{12}$$