1. (15 points) Compute

$$\begin{pmatrix} 1 & 3 & 0 \\ 2 & 5 & 7 \\ 3 & 6 & 0 \end{pmatrix} \cdot \begin{pmatrix} 3 & 4 & 4 & 2 \\ 5 & 5 & 6 & 7 \\ 6 & 8 & 9 & 10 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 18 & 19 & 22 & 23 \\ 73 & 89 & 101 & 109 \\ 39 & 42 & 48 & 48 \end{pmatrix}$$

2. (15 points) Show that isomorphism of simple graphs is an equivalent relation.

Solution: We need to verify the reflexivity, symmetry, and transitivity.

Reflexivity: use identity function as the isomorphism function.

Symmetry; If f is the isomorphism between G_1 and G_2. Then the inverse of f is the isomorphism between G_2 and G_1.

Transitivity: If f is the isomorphism between G_1 and G_2, and g is the isomorphism between G_2 and G_3, then gf is the isomorphism from G_1 to G_3.

Use the definition of isomorphism to check. If a and b are connected in G_1 then $f(a)$ and $f(b)$ must be connected in G_2, and then $g(f(a))$ and $g(f(b))$ must be connected in G_3. This shows G_1 and G_3 are isomorphic.

3. (15 points) Show that the property that a graph is bipartite is an isomorphic invariant.

Solution: If G and H are isomorphic and G is a bipartite graph, we show H is also a bipartite graph.

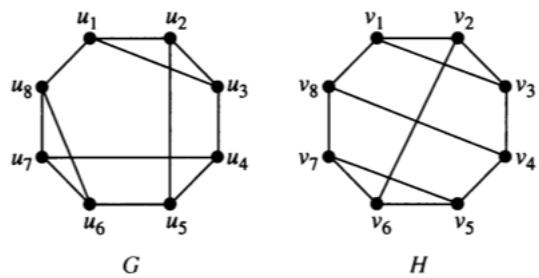
By definition, the vertex set of G can be divided into two disjoint subsets V_1 and V_2 such that each edge in G has an endpoint in V_1 and the other one in V_2.

Let f be the isomorphism function between G and H. Then let $W_1 = f(V_1)$ and $W_2 = f(V_2)$. As f is a bijective function, W_1 and W_2 are disjoint since V_1 and V_2 are. Also, $W_1 \cup W_2$ is the vertex set of H.

We only need to verify that every edge in H has an endpoint in W_1 and the other one in W_2. As G and H are isomorphism, then for every distinct vertices a and b in G, they are adjacent if and only if $f(a)$ and $f(b)$ are adjacent. Therefore, for any edge $e = \{a, b\}$ in G, we can find a corresponding one $e' = \{f(a), f(b)\}$ in H. As G

is bipartite, one of the vertices is in V_1 and the other one is in V_2, meaning one of $f(a)$ and $f(b)$ is in W_1 and the other one is in W_2. Therefore, H is bipartite.

4. (15 points) Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



Solution: G and H both have 8 vertices, 12 edges, and each vertex has degree 3. There exists circuit of length 3 to 8 in each graph. We will use a length 8 circuit to find the isomorphism.

Since the vertices have the same degree, the construction needs a little bit more observation. We realize that there exist two length 3 circuits in each graph which are connected by a single edge. For G, it is u_1, u_2, u_3 and u_8, u_7, u_6, the two are connected by edge $\{u_1, u_8\}$. For H, it is v_1, v_2, v_3 and v_5, v_6, v_7, the two are connected by edge $\{v_2, v_6\}$.

We use this particular information, build a length 8 circuit in G as

$$u_3, u_2, u_1, u_8, u_7, u_6, u_5, u_4.$$

The two length 3 circuits are traversed one after the other. This leads to a mapping

$$v_3, v_1, v_2, v_6, v_5, v_7, v_8, v_4.$$

With this particular order, we write down the adjacency matrices for each graph, and find they are the same as follows.

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

5. (20 points) Show that if a simple graph G has k connected components and these components have $n_1, n_2, \ldots, n_k$ vertices, respectively, then the number of edges of G does not exceed

$$\sum_{i=1}^{k} C(n_i, 2)$$

Solution: Each connect component with n_i vertices can have at most $C(n_i, 2)$ edges. That corresponds to the case every distinct vertices are connected with an edge. Since G is a simple graph, there can be at most one edge between any distinct vertices.

6. (15 points) (Bonus) Use previous result to show that a simple graph with n vertices and k connected components has at most $\frac{(n-k)(n-k+1)}{2}$ edges.
Hint: First, show that

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k),$$

where n_i is the number of vertices in the ith connected component. To show that, consider

$$\sum_{i=1}^{k} (n_i - 1) = n - k$$

Solution: Square on both sides of $\sum_{i=1}^{k}(n_i - 1) = n - k$, we then have

$$\sum_{i=1}^{k} (n_i - 1)^2 + \text{cross terms} = n^2 - 2nk + k^2$$

The left hand side can be simplified to

$$\sum_{i=1}^{k} (n_i - 1)^2 = \sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} 2n_i + k = \sum_{i=1}^{k} n_i^2 - 2n + k$$

Then, we have

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - 2nk + k^2 + 2n - k = n^2 - (k-1)(2n-k)$$

because all the cross terms are non-negative.

Then, use the result of # 5, we have the number of edges is at most

$$\sum_{i=1}^{k} C(n_i, 2) = \sum_{i=1}^{k} (n_i - 1)n_i/2 = \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{n}{2}$$

$$\leq \frac{n^2 - (k-1)(2n-k) - n}{2} = \frac{n^2 - 2nk + k^2 + n - k}{2} = \frac{(n-k)(n-k+1)}{2}$$

7. (20 points) Use previous result to show that a simple graph with n vertices is connected if if has more than $\frac{(n-1)(n-2)}{2}$ edges.

Solution: Notice that the value of $(n-k)(n-k+1)/2$ decreases when k becomes larger.

If a simple graph with n vertices is not connected, it will contain at least 2 connected components. Therefore, the value of k in previous problem is $k \geq 2$.

Then, there are at most $(n-2)(n-2+1)/2$ edges in the graph, which contradicts to the condition that the graph has more than $(n-1)(n-2)/2$ edges.

Therefore, the graph is connected.