Example
$L = \{a^nba^n \mid n > 0\}$

Closure Properties
A set is closed over an operation if

$L_1, L_2 \in \text{class}$
$L_1 \text{ op } L_2 = L_3$
$\Rightarrow L_3 \in \text{class}$

Example
$L_1 = \{x \mid x \text{ is a positive even integer}\}$
$L$ is closed under

addition?
multiplication?
subtraction?
division?

Closure of Regular Languages

Theorem 4.1 If $L_1$ and $L_2$ are regular languages, then

$L_1 \cup L_2$
$L_1 \cap L_2$
$L_1L_2$
$L_1^*$
$L_1^*$

are regular languages.
Proof (sketch)

$L_1$ and $L_2$ are regular languages
⇒ ∃ reg. expr. $r_1$ and $r_2$ s.t.
\[ L_1 = L(r_1) \text{ and } L_2 = L(r_2) \]
\[ r_1 + r_2 \text{ is r.e. denoting } L_1 \cup L_2 \]
⇒ closed under union
\[ r_1 r_2 \text{ is r.e. denoting } L_1 L_2 \]
⇒ closed under concatenation
\[ r_1^* \text{ is r.e. denoting } L_1^* \]
⇒ closed under star-closure

complementation:
$L_1$ is reg. lang.
⇒ ∃ DFA $M$ s.t. $L_1 = L(M)$
Construct $M'$ s.t.
final states in $M$ are
nonfinal states in $M'$
nonfinal states in $M$ are
final states in $M'$
⇒ closed under complementation

intersection:
$L_1$ and $L_2$ are reg. lang.
⇒ ∃ DFA $M_1$ and $M_2$ s.t.
\[ L_1 = L(M_1) \text{ and } L_2 = L(M_2) \]
$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$
$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$
Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$
\[ Q' = (Q \times P) \]
\[ \delta': \]
\[ \delta'((q_i, p_j), a) = (q_k, p_l) \text{ if } \]
\[ w \in L(M') \iff w \in L_1 \cap L_2 \]
⇒ closed under intersection
Example:

Regular languages are closed under

- reversal $L^R$
- difference $L_1 - L_2$
- right quotient $L_1 / L_2$
- homomorphism $h(L)$

**Right quotient**

Def: $L_1 / L_2 = \{ x | xy \in L_1 \text{ for some } y \in L_2 \}$

Example:

$L_1 = \{ a^* b^* \cup b^* a^* \}$
$L_2 = \{ b^n | n \text{ is even, } n > 0 \}$
$L_1 / L_2 = \emptyset$

**Theorem** If $L_1$ and $L_2$ are regular, then $L_1 / L_2$ is regular.

**Proof** (sketch)

$\exists$ DFA $M = (Q, \Sigma, \delta, q_0, F)$ s.t. $L_1 = L(M)$.

Construct DFA $M' = (Q, \Sigma, \delta, q_0, F')$

For each state $i$ do
  - Make $i$ the start state (representing $L_i'$)
  - if $L_i' \cap L_2 \neq \emptyset$ then
    - put $q_i$ in $F'$ in $M'$

QED.
Homomorphism

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$h: \Sigma \rightarrow \Gamma^*$$

Example:

$\Sigma = \{a, b, c\}$, $\Gamma = \{0, 1\}$

- $h(a) = 11$
- $h(b) = 00$
- $h(c) = 0$

$h(bc) =$

$h(ab^*) =$

Questions about regular languages:

$L$ is a regular language.

- Given $L$, $\Sigma$, $w \in \Sigma^*$, is $w \in L$?

- Is $L$ empty?

- Is $L$ infinite?

- Does $L_1 = L_2$?
Ch. 4.3 - Identifying Nonregular Languages

If a language $L$ is finite, is $L$ regular?

If $L$ is infinite, is $L$ regular?

- $L_1 = \{a^n b^m | n > 0, m > 0\} =$
- $L_2 = \{a^n b^n | n > 0\}$

Prove that $L_2 = \{a^n b^n | n > 0\}$ is ?

- Proof:
**Pumping Lemma:** Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

\[
\begin{align*}
|xy| & \leq m \\
|y| & \geq 1 \\
x_iy_i z & \in L \text{ for all } i \geq 0
\end{align*}
\]

**Meaning:** Every long string in $L$ (the constant $m$ above corresponds to the finite number of states in $M$ in the previous proof) can be partitioned into three parts such that the middle part can be “pumped” resulting in strings that must be in $L$.

**To Use the Pumping Lemma to prove $L$ is not regular:**

- **Proof by Contradiction.**
  - Assume $L$ is regular.
  - $\Rightarrow$ $L$ satisfies the pumping lemma.
  - Choose a long string $w$ in $L$, $|w| \geq m$. (The choice of the string is crucial. Must pick a string that will yield a contradiction).
  - Show that there is NO division of $w$ into $xyz$ (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $x_iy_i z \in L \forall i \geq 0$.
  - The pumping lemma does not hold. Contradiction!  
  - $\Rightarrow L$ is not regular. QED.

**Example** $L = \{a^n c b^n | n > 0 \}$

$L$ is not regular.

- **Proof:**
  - Assume $L$ is regular.
  - $\Rightarrow$ the pumping lemma holds.
  - Choose $w =$ where $m$ is the constant in the pumping lemma. (Note that $w$ must be chosen such that $|w| \geq m$.)
  - The only way to partition $w$ into three parts, $w = xyz$, is such that $x$ contains 0 or more $a$’s, $y$ contains 1 or more $a$’s, and $z$ contains 0 or more $a$’s concatenated with $c b^m$. This is because of the restrictions $|xy| \leq m$ and $|y| > 0$. So the partition is:

  It should be true that $x_iy_i z \in L$ for all $i \geq 0$. 
**Example** \( L = \{ a^n b^{n+s} c^s | n, s > 0 \} \)

\( L \) is not regular.

- **Proof:**
  Assume \( L \) is regular.
  \[ \Rightarrow \text{the pumping lemma holds.} \]
  Choose \( w = \)
  The only way to partition \( w \) into three parts, \( w = xyz \), is such that \( x \) contains 0 or more \( a \)'s, \( y \) contains 1 or more \( a \)'s, and \( z \) contains 0 or more \( a \)'s concatenated with the rest of the string \( b^{m+s}c^s \).
  This is because of the restrictions \( |xy| \leq m \) and \( |y| > 0 \). So the partition is:

**Example** \( \Sigma = \{ a, b \} \), \( L = \{ w \in \Sigma^* | n_a(w) > n_b(w) \} \)

\( L \) is not regular.

- **Proof:**
  Assume \( L \) is regular.
  \[ \Rightarrow \text{the pumping lemma holds.} \]
  Choose \( w = \)
  So the partition is:
**Example** \( L = \{ a^3 b^n c^{n-3} | n > 3 \} \)

\( L \) is not regular.

- **Proof:**
  
  Assume \( L \) is regular. \( \Rightarrow \) the pumping lemma holds.
  
  Choose \( w = a^3 b^m c^{m-3} \) where \( m \) is the constant in the pumping lemma. There are three ways to partition \( w \) into three parts, \( w = xyz \). 1) \( y \) contains only \( a \)'s 2) \( y \) contains only \( b \)'s and 3) \( y \) contains \( a \)'s and \( b \)'s
  
  We must show that each of these possible partitions lead to a contradiction. (Then, there would be no way to divide \( w \) into three parts s.t. the pumping lemma constraints were true).
  
  **Case 1:** (\( y \) contains only \( a \)'s). Then \( x \) contains 0 to 2 \( a \)'s, \( y \) contains 1 to 3 \( a \)'s, and \( z \) contains 0 to 2 \( a \)'s concatenated with the rest of the string \( b^m c^{m-3} \), such that there are exactly 3 \( a \)'s. So the partition is:
  
  \[
  x = a^k \quad y = a^j \quad z = a^{3-k-j} b^m c^{m-3}
  \]
  
  where \( k \geq 0, j > 0, \) and \( k + j \leq 3 \) for some constants \( k \) and \( j \).
  
  It should be true that \( xy^i z \in L \) for all \( i \geq 0 \).
  
  \( xy^2 z = (x)(y)(y)(z) = (a^k)(a^j)(a^j)(a^{3-j} b^m c^{m-3}) = a^{3+j} b^m c^{m-3} \notin L \) since \( j > 0, \) there are too many \( a \)'s. Contradiction!
  
  **Case 2:** (\( y \) contains only \( b \)'s) Then \( x \) contains 3 \( a \)'s followed by 0 or more \( b \)'s, \( y \) contains 1 to \( m-3 \) \( b \)'s, and \( z \) contains 3 to \( m-3 \) \( b \)'s concatenated with the rest of the string \( c^{m-3} \). So the partition is:
  
  \[
  x = a^3 b^k \quad y = b^j \quad z = b^{m-k-j} c^{m-3}
  \]
  
  where \( k \geq 0, j > 0, \) and \( k + j \leq m-3 \) for some constants \( k \) and \( j \).
  
  It should be true that \( xy^i z \in L \) for all \( i \geq 0 \).
  
  \( xy^2 z = a^3 b^k c^{m-3} \notin L \) since \( j > 0, \) there are too few \( b \)'s. Contradiction!
  
  **Case 3:** (\( y \) contains \( a \)'s and \( b \)'s) Then \( x \) contains 0 to 2 \( a \)'s, \( y \) contains 1 to 3 \( a \)'s, and 1 to \( m-3 \) \( b \)'s, \( z \) contains 3 to \( m-1 \) \( b \)'s concatenated with the rest of the string \( c^{m-3} \). So the partition is:
  
  \[
  x = a^{3-k} \quad y = a^k b^j \quad z = b^{m-j} c^{m-3}
  \]
  
  where \( 3 \geq k > 0, m-3 \geq j > 0 \) for some constants \( k \) and \( j \).
  
  It should be true that \( xy^i z \in L \) for all \( i \geq 0 \).
  
  \( xy^2 z = a^3 b^j a^k b^m c^{m-3} \notin L \) since \( j, k > 0, \) there are \( b \)'s before \( a \)'s. Contradiction!
  
  \( \Rightarrow \) There is no partition of \( w \).
  
  \( \Rightarrow \) \( L \) is not regular!. QED.
To Use Closure Properties to prove $L$ is not regular:

Using closure properties of regular languages, construct a language that should be regular, but for which you have already shown is not regular. Contradiction!

- **Proof Outline:**
  Assume $L$ is regular.
  Apply closure properties to $L$ and other regular languages, constructing $L'$ that you know is not regular.
  
  closure properties $\Rightarrow L'$ is regular.
  
  Contradiction!
  
  $L$ is not regular. QED.

**Example** $L=\{a^3b^nc^n-3 | n > 3\}$

$L$ is not regular.

- **Proof:** (proof by contradiction)
  Assume $L$ is regular.
  Define a homomorphism $h : \Sigma \to \Sigma^*$
  
  $h(a) = a$  \quad $h(b) = a$  \quad $h(c) = b$
  
  $h(L)$ =
Example \( L = \{ a^n b^m a^m | m \geq 0, n \geq 0 \} \)

\( L \) is not regular.

- **Proof:** (proof by contradiction)
  
  Assume \( L \) is regular.

---

Example: \( L_1 = \{ a^n b^n a^n | n > 0 \} \)

\( L_1 \) is not regular.

- **Proof:**
  
  Assume \( L_1 \) is regular.

  Goal is to try to construct \( \{ a^n b^n | n > 0 \} \) which we know is not regular.

  Let \( L_2 = \{ a^* \} \). \( L_2 \) is regular.

  By closure under right quotient, \( L_3 = L_1 \setminus L_2 = \{ a^n b^n a^p | 0 \leq p \leq n, n > 0 \} \) is regular.

  By closure under intersection, \( L_4 = L_3 \cap \{ a^* b^* \} = \{ a^n b^n | n > 0 \} \) is regular.

  Contradiction, already proved \( L_4 \) is not regular!

  Thus, \( L_1 \) is not regular. QED.