Uncertainty

CPS 170
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Why do we need uncertainty?

- Reason: Sh*t happens
- Actions don’t have deterministic outcomes

- Can logic be the “language” of AI???
- Problem: General logical statements are almost always false

- Truthful and accurate statements about the world would seem to require an endless list of qualifications
- How do you start a car?
- Call this “The Qualification Problem”
The Qualification Problem

• Is this a real concern?
• YES!
• Systems that try to avoid dealing with uncertainty tend to be brittle.
• Plans fail
• Finding shortest path to goal isn’t that great if the path doesn’t really get you to the goal

Probabilities

• Natural way to represent uncertainty

• People have intuitive notions about probabilities

• Many of these are wrong or inconsistent

• Most people don’t get what probabilities mean

• Finer details of this question still debated
Relative Frequencies

- Probabilities defined over events
- Space of all possible events is the “event space”

Think: Playing blindfolded darts with the Venn diagram..
- $P(A)$~percentage of dart throws that hit A (assuming a uniform distribution of dart hits over the area of the box)

Understanding Probabilities

- Initially, probabilities are “relative frequencies”
- This works well for dice and coin flips
- For more complicated events, this is problematic
- What is the probability that the democrats will control Congress in 2012?
  - This event only happens once
  - We can’t count frequencies
  - Still seems like a meaningful question
- In general, all events are unique
- “Reference Class” problem
Probabilities and Beliefs

• Suppose I have flipped a coin and hidden the outcome
• What is P(Heads)?

• Note that this is a statement about a belief, not a statement about the world
• The world is in exactly one state (at the macro level) and it is in that state with probability 1.
• Assigning truth values to probability statements is very tricky business
• Must reference speakers state of knowledge

Frequentism and Subjectivism

• Frequentists hold that probabilities must come from relative frequencies
• This is a purist viewpoint
• This is corrupted by the fact that relative frequencies are often unobtainable
• Often requires complicated and convoluted assumptions to come up with probabilities
• Subjectivists: probabilities are degrees of belief
  – Taints purity of probabilities
  – Often more practical
The Middle Ground

- No two events are ever identical, but
- No two events are ever totally unique either
- Probability that Obama will be elected in 2012?
  - He won once before
  - Conditions in next election will be similar, but not identical
  - Opponent will most likely be different

- In reality, we use probabilities as beliefs, but we allow data (relative frequencies) to influence these beliefs
- More precisely: We can use Bayes rule to combine our prior beliefs with new data

Why probabilities are good

- Subjectivists: probabilities are degrees of belief
- Are all degrees of belief probability?
  - AI has used many notions of belief:
    - Certainty Factors
    - Fuzzy Logic
- Can prove that a person who holds a system of beliefs inconsistent with probability theory can be tricked into accepted a sequence of bets that is guaranteed to lose (Dutch book)
So, what are probabilities mathematically?

- Probabilities are defined over random variables
- Random variables are usually represented with capitals: X, Y, Z
- Random variables take on values from a finite domain: d(X), d(Y), d(Z)
- We use lower case letters for values from domains

- X=x asserts: RV X has taken on value x
- P(x) is shorthand for P(X=x)

Event spaces for binary, discrete RVs

- 2 variable case

  ![Diagram](image)

- Important: Event space grows exponentially in number of random variables
- Components of event space = atomic events
Domains

- In the simplest case, domains are Boolean
- In general may include many different values
- Most general case: domains may be continuous
- This introduces some special complications

Kolmogorov’s axioms of probability

- $0 \leq P(a) \leq 1$
- $P(\text{true}) = 1; P(\text{false})=0$
- $P(a \text{ or } b) = P(a) + P(b) - P(a \text{ and } b)$
- Subtract to correct for double counting

- This is sufficient to completely specify probability theory for discrete variables
- Continuous variables need density functions
Atomic Events

• When several variables are involved, it is useful to think about atomic events
• An atomic event is a complete assignment to variables in the domain (compare with states in search)
• Atomic events are mutually exclusive
• Exhaust space of all possible events
• For n binary variables, how many unique atomic events are there?

Joint Distributions

• A joint distribution is an assignment of probabilities to every possible atomic event
• We can define all other probabilities in terms of the joint probabilities by marginalization:

\[ P(a) = P(a \land b) + P(a \land \neg b) \]

\[ P(a) = \sum_{e_i \in \varepsilon(a)} P(e_i) \]
Example

- \( P(\text{cold} \land \text{headache}) = 0.4 \)
- \( P(\neg \text{cold} \land \text{headache}) = 0.2 \)
- \( P(\text{cold} \land \neg \text{headache}) = 0.3 \)
- \( P(\neg \text{cold} \land \neg \text{headache}) = 0.1 \)

- What are \( P(\text{cold}) \) and \( P(\text{headache}) \)?

Independence

- If \( A \) and \( B \) are independent:
  \[
P(A \land B) = P(A)P(B)
  \]

- \( P(\text{cold} \land \text{headache}) = 0.4 \)
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- Are cold and headache independent?
Independence

• If A and B are mutually exclusive:
  \[ P(A \lor B) = P(A)+P(B) \]  (Why?)

• Examples of independent events:
  – Duke winning NCAA, Dem. winning white house
  – Two successive, fair coin flips
  – My car starting and my iPod working
  – etc.

• Can independent events be mutually exclusive?

Independence

• Convenient when it occurs, but don’t count on it
• When you have it:
  – \[ P(A \text{ and } B) = P(A)*P(B) \]
  – \[ P(A \text{ or } B) = P(A) + P(B) - P(A)P(B) \]

• Special cases: Disjoint events, perfectly correlated events
Why Probabilities Are Messy

- Probabilities are not truth-functional
- To compute $P(a \text{ and } b)$ we need to consult the joint distribution
  - sum out all of the other variables from the distribution
  - It is not a function of $P(a)$ and $P(b)$
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- This fact led to many approximations methods such as certainty factors and fuzzy logic (Why?)
- Neat vs. Scruffy...

The Scruffy Trap

- Reasoning about probabilities correctly requires knowledge of the joint distribution
- This is exponentially large
- Very convenient to assume independence
- Assuming independence (mutual exclusivity) when there is not independence (mutual exclusivity) leads to incorrect answers
- Examples:
  - ANDing symptoms
  - ORing symptoms
Conditional Probabilities

- Ordinary probabilities for random variables: *unconditional or prior probabilities*

- \( P(a \mid b) = \frac{P(a \text{ AND } b)}{P(b)} \)

- This tells us the probability of \( a \) given that we know only \( b \)

- If we know \( c \) and \( d \), we can’t use \( P(a \mid b) \) directly (without additional assumptions)

- Annoying, but solves the qualification problem...

Probability Solves the Qualification Problem

- \( P(\text{disease} \mid \text{symptom1}) \)

- This defines the probability of a disease given that we have observed only symptom1

- The conditioning bar indicates that the probability is defined with respect to a particular state of knowledge, *not as an absolute thing*
Condition with Bayes’s Rule

\[ P(A \land B) = P(B \land A) \]
\[ P(A \mid B)P(B) = P(B \mid A)P(A) \]
\[ P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} \]

Note that we will usually call Bayes’s rules “Bayes Rule”

Conditioning and Belief Update

• Suppose we know \( P(ABCDE) \)
• Observe \( B=b \), update our beliefs:

\[ P(ACDE \mid b) = \frac{P(ABCDE)}{P(b)} = \frac{P(ABCDE)}{\sum_{ACDE} P(ABcDE)} \]

Notation comment: This is a very condensed notation. \( P(ACDE \mid b) \) is not a number; it’s a distribution.
Example Revisited

- $P(\text{cold} \land \text{headache}) = 0.4$
- $P(\neg\text{cold} \land \text{headache}) = 0.2$
- $P(\text{cold} \land \neg\text{headache}) = 0.3$
- $P(\neg\text{cold} \land \neg\text{headache}) = 0.1$

- What is $P(\text{cold}|\text{headache})$?

Let’s Play Doctor

- Suppose $P(\text{cold}) = 0.7$, $P(\text{headache}) = 0.6$
- $P(\text{headache}|\text{cold}) = 0.57$
- What is $P(\text{cold}|\text{headache})$ using Bayes Rule?

$$P(c \mid h) = \frac{P(h \mid c)P(c)}{P(h)}$$

$$= \frac{0.57 \times 0.7}{0.6} = 0.66$$

- IMPORTANT: Not always symmetric
**Expectation**

- Most of us use expectation in some form when we compute averages
- What is the average value of a die roll?

\[ \frac{1+2+3+4+5+6}{6} = 3.5 \]

**Bias**

- What if not all events are equally likely?
- Suppose weighted die makes 6 2X more likely that anything else. What is average value of outcome?

\[ \frac{1 + 2 + 3 + 4 + 5 + 6}{7} = 3.86 \]

- Probs: 1/7 for 1...5, and 2/7 for 6

\[ (1 + 2 + 3 + 4 + 5) \times \frac{1}{7} + 6 \times \frac{2}{7} = 3.86 \]
Expectation in General

- Suppose we have some RV $X$
- Suppose we have some function $f(X)$
- What is the expected value of $f(X)$?

$$E f(x) = \sum_x P(X) f(X)$$

Sums of Expectations

- Suppose we have $f(X)$ and $g(Y)$.
- What is the expected value of $f(X) + g(Y)$?

$$E_x f(X) + g(Y) = \sum_x P(x \land y) (f(x) + g(y))$$

Linearity of Expectation

$$= \sum_x \sum_y P(x \land y) g(y) f(x) + \sum_x \sum_y P(x \land y) g(y)$$

$$= \sum_x f(x) \sum_y P(x \land y) + \sum_y g(y) \sum_x P(x \land y)$$

$$= \sum_x f(x) P(x) + \sum_y g(y) P(y)$$

$$= E_x f(x) + E_y g(y)$$
Continuous Random Variables

- Domain is some interval, region, or union of regions
- Uniform case: Simplest to visualize
  (event probability is proportional to area)
- Non-uniform case visualized with extra dimension

Gaussian (normal/bell) distribution:

![Gaussian Distribution Graph](image)

Updating Kolmogrov’s Axioms

- Use lower case for probability density
- Use end of the alphabet for continuous vars
- For discrete events: $0 \leq P(a) \leq 1$
- For densities: $0 \leq p(x)$

- Is $p(x) > 1$ possible???
Requirements on Continuous Distributions

- \( p(x) > 1 \) is possible so long as:
  \[
  \int_x p(x) \, dx = 1
  \]
- Don’t confuse \( p(x) \) and \( P(X=x) \)
- \( P(X=x) \) for any \( x \) is 0!

\[
P(x \in A) = \int_A p(x) \, dx
\]

Cumulative Distributions

- When distribution is over numbers, we can ask:
  - \( P(X>=c) \) for some \( c \)
  - \( P(X<c) \) for some \( c \)
  - \( P(a<=X<=b) \) for some, \( a \) and \( b \)
- Solve by
  - Summation
  - Integration
- Cumulative sometimes called
  - CDF
  - Distribution function
Sloppy Comment about Continuous Distributions

- In many, many cases, you can generalize what you know about discrete distributions to continuous distributions, replacing “P” with “p” and “Σ” with “∫”

- Proper treatment of this topic requires measure theory and is beyond the scope of the text and class

Probability Conclusions

- Probabilistic reasoning has many advantages:
  - Solves qualification problem
  - Is better than any other system of beliefs (Dutch book argument)

- Probabilistic reasoning is tricky
  - Some things decompose nicely: linearity of expectation, conjunctions of independent events, disjunctions of disjoint events
  - Some things can be counterintuitive at first: conjunctions of arbitrary events, conditional probability

- Reasoning efficiently with probabilities poses significant data structure and algorithmic challenges for AI

  (Roughly speaking, the AI community realized some time around 1990 that probabilities were the right thing and has spent the last 20 years grappling with this realization.)