NP Hardness/Completeness Overview

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CPS 170

Digression: NP-Hardness

- NP hardness is not an AI topic
- You will not be tested on it explicitly, but

- It's important for all computer scientists
- Understanding it will deepen your understanding of AI (and other CS) topics
- You will be expected to understand its relevance and use for AI problems

- Eat your vegetables; they're good for you

P and NP

- P and NP are about decision problems
- P is set of problems that can be solved in polynomial time
- NP is a superset of P
- NP is the set of problems that:
  - Have solutions which can be verified in polynomial time or, equivalently,
  - can be solved by a non-deterministic Turing machine in polynomial time (OK if you don't know what that means yet)
- Roughly speaking:
  - Problems in P are tractable – can be solved in a reasonable amount of time, and Moore's law helps
  - Problems in NP *might* not be tractable

NP-hardness

- Many problems in AI are NP-hard (or worse)
- What does this mean?
- These are some of the hardest problems in CS
- Identifying a problem as NP hard means:
  - You probably shouldn't waste time trying to find a polynomial time solution
  - If you find a polynomial time solution, either
    - You have a bug
    - Find a place on your shelf for your Turing award
- NP hardness is a major triumph (and failure) for computer science theory

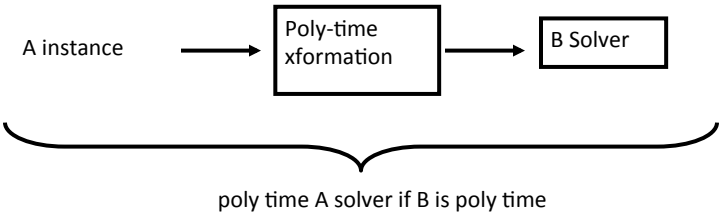
Understanding the class NP

- A class of *decision problems* (Yes/No)
- Solutions can be verified in polynomial time
- Examples:
  - Graph coloring:

  - Sortedness: [1 2 3 4 5 8 7]

What is NP hardness?

- An NP hard problem is at least has hard as the hardest problems in NP
- The hardest problems in NP are *NP-complete*
- Demonstrate hardness via *reduction*
  - Use one problem to solve another
  - A is reduced to B, if we can use B to solve A:



Hardness vs. Completeness

- For something to be NP-*complete*, must be in NP
- If something is NP-*hard*, it ***could be even harder*** than the hardest problems in NP

- Proving completeness is stronger theoretical result – says more about the problem

Why care about NP-completeness?

- Solving any one NP-complete problem gives you the key to all others
- All NP-complete problems are, in a sense, *equivalent*
- Insight into solving any one gives you insight into solving a vast array of problems of extraordinary practical and economic significance

The First NP Complete Problem (Cook 1971)

- SAT:

$$(X_1 \vee \bar{X}_7 \vee X_{13}) \wedge (\bar{X}_2 \vee X_{12} \vee X_{25}) \wedge \ldots$$

- Want to find an assignment to all variables that makes this expression evaluate to true
- NP-complete for clauses of size 3 or greater
- How would you prove this?

Hardness w/o completeness?

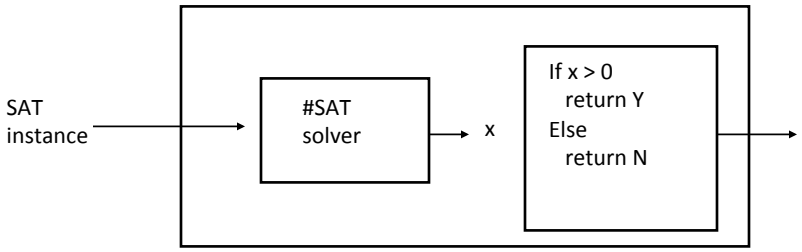
- NP hardness is a weaker claim (says less about the problem) than NP completeness, but
- NP hard problems might be harder than NP-complete
- NP hard if an NP complete problem is reducible to it
- NP completeness = NP hardness + NP membership
- Consider the problem #SAT
  - How many satisfying assignments to:

$$(X_1 \vee \bar{X}_7 \vee X_{13}) \wedge (\bar{X}_2 \vee X_{12} \vee X_{25}) \wedge \ldots$$

  - Is this in NP? (Not even a decision problem)
  - Is it NP-hard?

#SAT is NP-hard

- Theorem: #SAT is NP hard
- Proof:
  - Reduce SAT to #SAT



P=NP?

- Biggest open question in CS
- Can NP-complete problems be solved in polynomial time?
- Probably not, but nobody has been able to prove it yet
- Recent attempt at proof detailed in NY Times, one of many false starts: http://www.nytimes.com/2009/10/08/science/Wpolynom.html

How challenging is "P=NP?"

- Princeton University CS department
- See: http://www.cs.princeton.edu/general/bricks.php
- Photo from: http://stuckinthebubble.blogspot.com/2009/07/three-interesting-points-on-princeton.html

How To Avoid Embarrassing Yourself

- Don't say: "I proved that it requires exponential time."
  if you really meant:
- "I proved it's NP-Hard/Complete"

- Don't say: "The problem is NP" (which doesn't even make sense)
  if you really meant:
- "The problem NP-Hard/Complete"

- Don't reduce new problems to NP-hard complete problems if you meant to prove the new problem is hard
- Such a reduction is backwards. What you really proved is that you can use a hard problem to solve an easy one. Always think carefully about the direction of your reductions

NP-Completeness Summary

- NP-completeness tells us that a problem belongs to class of similar, hard problems.
- What if you find that a problem is NP hard?
  - Look for good approximations
  - Find different measures of complexity
  - Look for tractable subclasses
  - Use heuristics