Brief Comments on Game Theory

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CPS 170

What is Game Theory

- Very general mathematical framework to study situations where multiple agents interact, including:
  - Popular notions of games
  - Everything up to and including multistep, multiagent, simultaneous move, partial information games
  - Can even including negotiating, posturing and uncertainty about the players and game itself
- von Neumann and Morgenstern (1944) was a major launching point for modern game theory
- Nash: Existence of equilibria in general sum games

Covered Today

- 2 player, zero sum simultaneous move games
- Example: Rock, Paper, Scissors

- Linear programming solution

Linear Programs (max formulation)

$$\begin{aligned} \text{maximize}: & \; c^T x \\ \text{subject to}: & \; \mathbf{A}x \leq b \\ : & \; x \geq 0 \end{aligned}$$

- Note: min formulation also possible
  - Min: $c^T x$
  - Subject to: $Ax \geq b$
- Some use equality as the canonical representation (introducing slack variables)
- LP tricks
  - Multiply by -1 to reverse inequalities
  - Can easily introduce equality constraints, or arbitrary domain constraints

Rock, Paper, Scissors Zero Sum Formulation

- In zero sum games, one player's loss is other's gain
- Payoff matrix:

	R	*P*	*S*
R	0	–1	1
P	1	0	–1
S	–1	1	0

- Minimax solution maximizes worst case outcome

Rock, Paper, Scissors Equations

- R,P,S = probability that we play rock, paper, or scissors respectively (R+P+S = 1)
- U is our expected utility
- Bounding our utility:
  - Opponent rock case: $U \le P - S$
  - Opponent paper case: $U \le S - R$
  - Opponent scissors case: $U \le R - P$
- Want to maximize U subject to constraints
- Solution: (1/3, 1/3, 1/3)

Rock, Paper, Scissors LP Formulation

- Our variables are: $x=[U,R,P,S]^T$
- We want:
  - Maximize U
  - $U \leq P - S$
  - $U \leq S - R$
  - $U \leq R - P$
  - $R+P+S = 1$
- How do we make this fit:

$$\boxed{\begin{aligned} &\text{maximize}: c^T x \\ &\text{subject to}: \mathbf{A}x \leq b \\ &\qquad\qquad : x \geq 0 \end{aligned}}\ ?$$

Rock Paper Scissors LP Formulation

$$x = [U,R,P,S]^T$$

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{pmatrix}$$

$$b = [0,0,0,1,-1]^T$$

$$c = [1,0,0,0]^T$$

$$\boxed{\begin{aligned} &\text{maximize}: c^T x \\ &\text{subject to}: \mathbf{A}x \leq b \\ &\qquad\qquad : x \geq 0 \end{aligned}}$$

Rock, Paper, Scissors Solution

- If we feed this LP to an LP solver we get:
  - R=P=S=1/3
  - U=0
- Solution for the other player is:
  - The same…
  - By symmetry
- This is the minimax solution
- This is also an equilibrium
  - No player has an incentive to deviate
  - (Defined more precisely later in the course)

Tangent: Why is RPS Fun?

- OK, it's not…
- Why *might* RPS be fun?
  - Try to exploit non-randomness in your friends
  - Try to be random yourself

Minimax Solutions in General

- Minimax solutions for 2-player zero-sum games can always be found by solving a linear program
- The minimax solutions will also be equilibria

- For general sum games:
  - Minimax does not apply
  - Equilibria may not be unique
  - Need to search for equilibria using more computationally intensive methods