CPS 173

Linear programming, integer linear programming, mixed integer linear programming

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Example linear program

- We make reproductions of two paintings

  \[
  \begin{align*}
  \text{maximize} & \quad 3x + 2y \\
  \text{subject to} & \quad 4x + 2y \leq 16 \\
  & \quad x + 2y \leq 8 \\
  & \quad x + y \leq 5 \\
  & \quad x \geq 0 \\
  & \quad y \geq 0
  \end{align*}
  \]

- Painting 1 sells for $30, painting 2 sells for $20
- Painting 1 requires 4 units of blue, 1 green, 1 red
- Painting 2 requires 2 blue, 2 green, 1 red
- We have 16 units blue, 8 green, 5 red
Solving the linear program graphically

maximize $3x + 2y$

subject to

$4x + 2y \leq 16$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

optimal solution: $x=3$, $y=2$
Proving optimality

\[ \text{maximize } 3x + 2y \]
\[ \text{subject to } \]
\[ 4x + 2y \leq 16 \]
\[ x + 2y \leq 8 \]
\[ x + y \leq 5 \]
\[ x \geq 0 \]
\[ y \geq 0 \]

Recall: optimal solution:

\[ x=3, \ y=2 \]

Solution value = 9+4 = 13

How do we prove this is optimal (without the picture)?
Proving optimality…

maximize \( 3x + 2y \)

subject to

\[
\begin{align*}
4x + 2y &\leq 16 \\
x + 2y &\leq 8 \\
x + y &\leq 5 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]

We can rewrite the blue constraint as

\[
2x + y \leq 8
\]

If we add the red constraint

\[
x + y \leq 5
\]

we get

\[
3x + 2y \leq 13
\]

Matching upper bound!

(Really, we added .5 times the blue constraint to 1 times the red constraint)
Linear combinations of constraints

maximize $3x + 2y$

subject to

$4x + 2y \leq 16$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

$b(4x + 2y \leq 16) +
\quad g(x + 2y \leq 8) +
\quad r(x + y \leq 5) =

(4b + g + r)x +
\quad (2b + 2g + r)y \leq
\quad 16b + 8g + 5r$

$4b + g + r$ must be at least 3

$2b + 2g + r$ must be at least 2

Given this, minimize $16b + 8g + 5r$
Using LP for getting the best upper bound on an LP

\[
\begin{align*}
\text{maximize} & \quad 3x + 2y \\
\text{subject to} & \quad 4x + 2y \leq 16 \\
& \quad x + 2y \leq 8 \\
& \quad x + y \leq 5 \\
& \quad x \geq 0 \\
& \quad y \geq 0 \\
\text{minimize} & \quad 16b + 8g + 5r \\
\text{subject to} & \quad 4b + g + r \geq 3 \\
& \quad 2b + 2g + r \geq 2 \\
& \quad b \geq 0 \\
& \quad g \geq 0 \\
& \quad r \geq 0
\end{align*}
\]

the dual of the original program

• Duality theorem: any linear program has the same optimal value as its dual!
Modified LP

maximize $3x + 2y$
subject to
$4x + 2y \leq 15$
$x + 2y \leq 8$
$x + y \leq 5$
$x \geq 0$
y $\geq 0$

Optimal solution: $x = 2.5,$ $y = 2.5$
Solution value = $7.5 + 5 = 12.5$

Half paintings?
Integer (linear) program

maximize \( 3x + 2y \)
subject to
\( 4x + 2y \leq 15 \)
\( x + 2y \leq 8 \)
\( x + y \leq 5 \)
\( x \geq 0, \text{ integer} \)
\( y \geq 0, \text{ integer} \)

optimal IP solution: \( x=2, y=3 \) (objective 12)

optimal LP solution: \( x=2.5, y=2.5 \) (objective 12.5)
Mixed integer (linear) program

 maximize $3x + 2y$
 subject to
 $4x + 2y \leq 15$
 $x + 2y \leq 8$
 $x + y \leq 5$
 $x \geq 0$
 $y \geq 0$, integer

optimal IP solution: $x=2$, $y=3$ (objective 12)
optimal LP solution: $x=2.5$, $y=2.5$ (objective 12.5)
optimal MIP solution: $x=2.75$, $y=2$ (objective 12.25)
Exercise in modeling: knapsack-type problem

- We arrive in a room full of precious objects
- Can carry only 30kg out of the room
- Can carry only 20 liters out of the room
- Want to maximize our total value
- Unit of object A: 16kg, 3 liters, sells for $11
  - There are 3 units available
- Unit of object B: 4kg, 4 liters, sells for $4
  - There are 4 units available
- Unit of object C: 6kg, 3 liters, sells for $9
  - Only 1 unit available
- What should we take?
The general version of this knapsack LP

\[
\text{maximize } \sum_j p_j x_j \\
\text{subject to } \sum_j w_j x_j \leq W \\
\sum_j v_j x_j \leq V \\
(x \text{ for all } j) \ x_j \leq a_j \\
(x \text{ for all } j) \ x_j \geq 0, \ x_j \text{ integer}
\]
Exercise in modeling: cell phones (set cover)

- We want to have a working phone in every continent (besides Antarctica)
- … but we want to have as few phones as possible
- Phone A works in NA, SA, Af
- Phone B works in E, Af, As
- Phone C works in NA, Au, E
- Phone D works in SA, As, E
- Phone E works in Af, As, Au
- Phone F works in NA, E
Exercise in modeling: hot-dog stands

- We have two hot-dog stands to be placed in somewhere along the beach
- We know where the people that like hot dogs are, how far they are willing to walk
- Where do we put our stands to maximize #hot dogs sold? (price is fixed)