Due: Thursday, March 22. Submissions should be typeset (LaTeX preferred). Explain the working of each algorithm simply and precisely. The correctness and running time of all algorithms must be backed up by formal arguments.

Question 1 [20 points] The reverse of a directed graph $G = (V, E)$ is another directed graph $G^R = (V, E^R)$ on the same vertex set, but with all edges reversed; that is, $E^R = \{(v, u) : (u, v) \in E\}$. Give a linear-time algorithm for computing the reverse of a graph in adjacency list format.

Question 2 [20 points] A bipartite graph is a graph $G = (V, E)$ whose vertices can be partitioned into two sets ($V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$) such that there are no edges between vertices in the same set (for instance, if $u, v \in V_1$, then there is no edge between $u$ and $v$).

(a) Give a linear-time algorithm to determine whether an undirected graph is bipartite.
(b) There are many other ways to formulate this property. For instance, an undirected graph is bipartite if and only if it can be colored with just two colors. Prove the following formulation: an undirected graph is bipartite if and only if it contains no cycles of odd length.
(c) At most how many colors are needed to color in an undirected graph with exactly one odd length cycle?

Question 3 [20 points] Suppose a CS curriculum consists of $n$ courses, all of them mandatory. The prerequisite graph $G$ has a node for each course, and an edge from course $v$ to course $w$ if and only if $v$ is a prerequisite for $w$. Find an algorithm that works directly with this graph representation, and computes the minimum number of semesters necessary to complete the curriculum (assume that a student can take any number of courses in one semester). The running time of your algorithm should be linear.

Question 4 [20 points] Give a linear-time algorithm to find an odd-length cycle in a directed graph. You may NOT assume that the graph is strongly connected. (Hint: First solve this problem under the assumption that the graph is strongly connected. Then show that every directed graph is a DAG of its strongly connected components.)

Question 5 [20 points] Let $G = (V, E)$ be an undirected graph. Prove that if all its edge weights are distinct, then it has a unique minimum spanning tree.