Question 1 [20 points] There is a network of roads $G = (V, E)$ connecting a set of cities $V$. Each road in $E$ has an associated length $l_e$. There is a proposal to add one new road to this network, and there is a list $E'$ of pairs of cities between which the new road can be built. Each such potential road $e' \in E'$ has an associated length. As a designer for the public works department you are asked to determine the road $e' \in E'$ whose addition to the existing network $G$ would result in the maximum decrease in the driving distance between two fixed cities $s$ and $t$ in the network. Give an efficient algorithm for solving this problem.

Question 2 [20 points] Often there are multiple shortest paths between two nodes of a graph. Give a linear-time algorithm for the following task.
Input: Undirected graph $G = (V, E)$ with unit edge lengths; nodes $u, v \in V$.
Output: The number of distinct shortest paths from $u$ to $v$.

Question 3 [20 points] You are given a directed graph $G(V, E)$. Each edge $e \in E$ is associated with two lengths $l_1(e)$ and $l_2(e)$, which are non-negative integers in the range $\{0, 1, \ldots, M\}$. Given two vertices $s, t \in V$ and two length bounds $A$ and $B$, give an efficient algorithm to decide if there is a path from $s$ to $t$ whose length according to the function $l_1$ is at most $A$ and whose length according to the function $l_2$ is at most $B$. What is the running time of your algorithm?

Question 4 [20 points] There are many common variations of the maximum flow problem. Here are two of them.
(a) There are many sources and many sinks, and we wish to maximize the total flow from all sources to all sinks.
(b) Each vertex also has a capacity on the maximum flow that can enter it.
Both of these can be solved efficiently. Show this by reducing (a) and (b) to the original max-flow problem.

Question 5 [20 points] Shortest path algorithms can be applied in currency trading. Let $c_1, c_2, \ldots, c_n$ be various currencies, for instance, $c_1$ might be dollars, $c_2$ pounds, and $c_3$ lire. For any two currencies $c_i$ and $c_j$, there is an exchange rate $r_{ij}$, this means that you can purchase $r_{ij}$ units of currency $c_j$ in exchange for one unit of $c_i$. These exchange rates satisfy the condition that $r_{ij} \times r_{ji} < 1$, so that if you start with a unit of currency $c_i$, change it into currency $c_j$ and then convert back to currency $c_i$, you end up with less than one unit of currency $c_i$ (the difference is the cost of the transaction).
(a) Give an efficient algorithm for the following problem: Given a set of exchange rates $r_{ij}$, and two currencies $s$ and $t$, find the most advantageous sequence of currency exchanges for converting currency $s$ into currency $t$. Toward this goal, you should represent the currencies
and rates by a graph whose edge lengths are real numbers. The exchange rates are updated frequently, reflecting the demand and supply of the various currencies. Occasionally the exchange rates satisfy the following property: there is a sequence of currencies $c_{i_1}, c_{i_2}, \ldots, c_{i_k}$ such that $r_{i_1,i_2} \times r_{i_2,i_3} \times \cdots \times r_{i_{k-1},i_k} \times r_{i_k,i_1} > 1$. This means that by starting with a unit of currency $c_{i_1}$ and then successively converting it to currencies $c_{i_2}, c_{i_3}, \ldots, c_{i_k}$, and finally back to $c_{i_1}$, you would end up with more than one unit of currency $c_{i_1}$. Such anomalies last only a fraction of a minute on the currency exchange, but they provide an opportunity for risk-free profits.

(b) Give an efficient algorithm for detecting the presence of such an anomaly. Use the graph representation you found above.