$r$-Permutation with Repetition:

- Choosing $r$ elements from a set of $n$ elements with repetition allowed.

Example: $S = \{a, \ldots, z\}$. So $181 = 26$

5-letter words with repetition of letters allowed

\[
\begin{align*}
\text{Number of ways} &= 26 \times 26 \times 26 \times 26 \times 26 \\
&= 26^5
\end{align*}
\]

Theorem: The number of $r$-permutations of a set of $n$ objects with repetition allowed is $n^r$.

Proof: $n$ ways to choose the first object, $n$ ways to choose the second, and so on. Thus,

\[
\text{Total ways} = n^r.
\]
Permutations with Distinguishable Objects:

Example: How many different things can be made by reordering the word \textit{SUCCESS}?

Solution:
• Distinct letters \textit{SUCCE}
  \textit{D} 3 1 2 !

• S can be placed in any of 7 positions in \( \text{C}(7,3) \) ways.
• C can be placed in remaining 4 positions in \( \text{C}(4,2) \) ways.
• U can be placed in any of 3 positions in \( \text{C}(3,1) \) ways.
• E can be placed in any of 1 positions in \( \text{C}(1,1) \) ways.

\[ \# \text{different strings} = \text{C}(7,3) \times \text{C}(4,2) \times \text{C}(3,1) \times \text{C}(1,1) = 420 \]  

Theorem: Let \( A \) be the set \( \{a_1, \ldots, a_n\} \), and let \( r_1, r_2, \ldots, r_n \) be non-negative integers. The number of permutations of the set \( A \) where each element \( a_i \) is repeated exactly \( r_i \) times is:

\[ \frac{(r_1 + r_2 + \ldots + r_n)!}{r_1! \, r_2! \cdots r_n!} \]

Proof:
• Let \( N = r_1 + r_2 + \ldots + r_n \).
• Here: cross the subscripts of $a_1$.

• This defines a 1-to-1 mapping from old permutations to permutations where $a_1$ is repeated $\tau_1$ times and all others are repeated once.

$\therefore \# \text{ new permutations} = \frac{N!}{\tau_1!}$

• Continue this way for $a_2, \ldots, a_n$, we will finally obtain the theorem proof!

• Alternatively, by previous example, we can write formally, the following:

$$\# \text{ ways} = C(N, \tau_1) \times C(N \tau_1, \tau_2) \times C(N - \tau_1 - \tau_2, \tau_3) \times \cdots \times C(N - \tau_1 - \tau_2 - \cdots - \tau_{n-1}, \tau_n)$$

$$= \frac{N!}{\tau_1! \tau_2! \cdots \tau_n!} = \frac{(\tau_1 + \tau_2 + \cdots + \tau_n)!}{\tau_1! \tau_2! \cdots \tau_n!}$$
Distributing Objects into Boxes:

- **Distinguishable objects in distinguishable boxes:**
  
  **Theorem:** The number of ways to distribute \( n \) distinguishable objects to \( k \) boxes so that \( n_i \) objects are placed into box \( i \), \( i = 1, 2, \ldots, k \) equals
  
  \[
  \frac{n!}{n_1! n_2! \cdots n_k!}
  \]
  
  - **Example:** How many ways are there to distribute a deck of 52 cards to each of 4 players from the standard deck of 52 cards?
  
  - **Solution:**
    
    \[
    \frac{52!}{5! 5! 5! 5! 32!} \quad (= \text{by product rule } C(52, 5) \times C(47, 5) \times C(42, 5) \times C(37, 5))
    \]

- **Distinguishable Objects and Distinguishable Boxes:**

  - \# ways of placing \( n \) indistinguishable objects into \( k \) distinguishable boxes = \# ways of \( \binom{n}{k} \) combinations of
    
    = the number of \( n \)-combinating of a set with \( k \) elements when repetition is allowed.

  - **Example:** \# ways of placing 10 distinguishable balls into 8 distinguishable boxes = \# 10-combinating from a set of 8 elelents with repetition allowed = \( C(10+8-1, 10) = C(17, 10) = 17! / 10! 7! \)
Distinguishable Objects and Indistinguishable Boxes:

- $n$ distinguishable objects
- $k$ indistinguishable boxes.

**Hard Problem!**

$$\text{#ways} = \sum_{j=1}^{k} S(n,j)$$

where

$$S(n,j) = \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (j-i)^n.$$  

**Stirling numbers of the second kind.**

- Usual practical way to solve this is to enumerate all ways for a reasonably small sized instance of the problem.

**Example:** Count the number of ways to pack six copies of the same book into four identical boxes, where a box can contain as many as six books.

**Solution:** Enumerate the possibilities in the decreasing # books in a box:

- \[ \begin{array}{cccc}
       & 6 & 1 & 1 \\
       5 & 4 & 2 & 1 \\
       4 & 3 & 3 & 1 \\
       3 & 3 & 2 & 2 \\
       2 & 2 & 2 & 2 \\
       2 & 2 & 1 & 1 \\
     \end{array} \]

9 ways!

**Partition!**
Theorem: The number of \( r \)-combinations with repetition of an \( n \)-element set is: \( \binom{n+r-1}{r} \).

Proof:
- Let \( S \) = set of \( n \) elements ordered in some way.
- We will give a bijection between \( r \)-combinations with repetition of the set \( S \) and strings of \( \text{stars} + \text{bars} \).
- Let \( R \) = a particular \( r \)-combination with repetition of \( S \).
- \( n-1 \) bars and \( r \) stars are used.

\[
\begin{array}{cccc}
\text{\_} & \text{\_} & \cdots & \text{\_} \\
& n-1 \text{ bars mark } n \text{ cells/regions}
\end{array}
\]

- Add a * to recognize, each time the \( i \)-th element of \( S \) appears in \( R \).
- What do we get here? \( r \)-combination \( \mapsto \) \( r \) *'s and \( n-r \) |'s.

Example: \( S = \{A, B, C, D, E\} \rightarrow R = 7 \)-combination with repetition \( \{A, B, B, B, D, E, E\} \). The \( * \rightarrow \) map is:

\[
\begin{array}{l}
* & * & * * * & * & * & * \\
A & B & B & B & D & E & E
\end{array}
\]
• Note since the mapping is a bijection, it has no reverse, i.e., if we can count the #storage containing \((n-1)\) bars and \(\&\) stars, we of course have the count for \#r-combinations with repetition of \(n\)-element set with repetition.

• #storage with \((n-1)\) bars and \(r\) stars

\[= \text{#ways to choose } r \text{ distinct positions for the stars in a string of } n+r-1 \text{ stars and bars} \]

\[= \text{ordinary } r\text{-combination of a set with } n+r-1 \text{ elements} \]

\[= \binom{n+r-1}{r}. \]

\[\square\]

• Example: we want to buy \(12\) cookies and there are \(21\) varieties available. How many options do we have?

Solution:

• Observe: Repetition is allowed.

• We seek \(12\)-combinations in a set of \(21\) elements.

\[\therefore \text{#ways} = \binom{12+21-1}{12} = \binom{32}{12}.\]
Example: How many solutions does the equation
\[ x_1 + x_2 + x_3 = 11 \]
have where \( x_1, x_2, x_3 \) are nonnegative integers?

Solution:

- Solution = a way of selecting 11 items from a set with 3 elements \( \{ x_1, x_2, x_3 \} \) where \( x_i \) denote a "type" here.
- \# Solutions = \#11-combinations with repetitions from a set with 3 elements

\[
= C(3 + 11 - 1, 11) = C(13, 11) = C(13, 2) = \frac{13!}{11!2!} = 78.
\]

Balls and Bins: (aka: Indistinguishable Objects & Distinguishable Boxes)

- \( n \) identical balls
- \( n \) identical boxes

(Mini) Theorem: There are \( \binom{n + r - 1}{r} \) ways of arranging \( r \) identical balls into \( n \) distinct boxes.
Theorem: The number of $r$-combinations with repetitions of an $n$-element set that contains every element in the set at least once is; $$\binom{n-1}{r-1}.$$

Proof: Consider an $r$-combination.

- It contains the entire $n$-element set together with an $(r-n)$-combination with repetitions of the $n$-element set.
- The number of such combinations is;

$$\binom{n + (r-n)-1}{r-n} = \binom{r-1}{r-n} = \binom{n-1}{n-1}.$$  

Example: Given balls of 3 colors: RED, BLUE, GREEN. How many ways are there to choose five balls so that we get at least one ball from each color?

Solution: $r = 5, n = 3$. #ways = $\binom{3-1}{5-1} = 6$. 
### Summary:

<table>
<thead>
<tr>
<th>Type</th>
<th>Repetitions Allowed</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )-permutation</td>
<td>NO</td>
</tr>
<tr>
<td>( r )-combination</td>
<td>NO</td>
</tr>
<tr>
<td>( r )-permutations</td>
<td>YES</td>
</tr>
<tr>
<td>( r )-combinations</td>
<td>YES</td>
</tr>
</tbody>
</table>

### Formula

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

\[
\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}
\]

### Type

- Distinguishable balls & Distinguishable boys
- Distinguishable balls & Indistinguishable boys
- Distinguishable balls & Distinguishable boys
- Distinguishable balls & Indistinguishable boys
- Indistinguishable balls & Distinguishable boys
- Indistinguishable balls & Indistinguishable boys
- Distinguishable balls & Distinguishable boys
- Distinguishable balls & Indistinguishable boys

**Formula**

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

\[
\binom{n+1}{r} = \frac{(n+1)!}{r!(n+1-r)!}
\]

**Hard!**

Enumerate all possibilities by inspection!