Due Date: April 9, 2013

Problem 1: A subsequence is obtained by extracting a subset of elements from a sequence, but keeping them in the same order; the elements of the subsequence need not be contiguous in the original sequence.

Call a sequence $X[1..k]$ of numbers *oscillating* if $X[i] < X[i+1]$ for all even i, and $X[i] > X[i+1]$ for all odd i. Describe an efficient algorithm to compute the length of the longest oscillating subsequence of an arbitrary array A of n integers.

Problem 2: Suppose we have a text consisting of a sequence of *words*, $W = w_1 w_2 \ldots w_n$, where w_i consists of c_i characters. A *formatting* of W consists of a partition of the words in W into *lines*. In the words assigned to a single line, there should be a space after each word except the last; and the maximum line length is L. Thus, if $w_j, w_{j+1}, \ldots, w_k$ are assigned to one line, then we should have

$$\left[\sum_{i=j}^{k-1}(c_i + 1)\right] + c_k \leq L$$

We will call an assignment of words to a line *valid* if it satisfies this inequality. The difference between the left-hand side and the right-hand side is called the *slack* of the line, that is, the number of spaces left at the right margin.

Give an efficient algorithm that finds a partition of a set of words W into valid lines, so that the sum of the **squares** of the slacks of all lines (including the last line) is minimized.

Problem 3: Integer linear programming (ILP) is the same as linear programming except that one considers only integer solutions, i.e.,

$$\begin{aligned} \min c^T x \qquad & \text{s.t.} \\ Ax \leq b & \\ x \geq 0, x \in \mathbb{Z} & \end{aligned}$$

The vertex cover of a graph $G = (V, E)$ is a subset $C \subseteq V$ of vertices so that each edge in E is incident to at least one of the vertices in C.

(i) Show that the problem of computing the minimum-size vertex cover can be formulated as an instance of ILP.

(ii) Suppose we relax the ILP in (i) to an LP by removing the integer constraint $x \in \mathbb{Z}$. Show that there is a graph for which the optimal LP solution is not integral (i.e., values of some of the variables are not integers).

Problem 4: Write the dual to the following linear program.

$$\begin{aligned} \max x_1 + x_2 \\ 2x_1 + x_2 \leq 3 \\ x_1 + 3x_2 \leq 5 \\ x_1, x_2 \geq 0 \end{aligned}$$

Find the optimal solutions to both primal and dual LPs using the simplex method. For the primal, you can use $x_1 = 0$ and $x_2 = 0$ as the initial basic feasible solution (BFS), and for the dual, you can use $y_1 = 1$ and $y_2 = 0$ as the initial BFS, where y_1 and y_2 are the dual variables associated with the first and second constraints in the primal, respectively.

Problem 5: (**Matching pennies**) In this simple two-player game, the players (call them R and C) each choose an outcome, *heads* or *tails*. If both outcomes are equal, C gives a dollar to R; if the outcomes are different, R gives a dollar to C.

(i) Represent the payoffs by a 2×2 matrix.

(ii) What is the value of this game? Compute the optimal strategies for the two players using the simplex algorithm.