Methods for Transforming Grammars (Read Ch 6 in Linz Book)

We will consider CFL without \( \lambda \). It would be easy to add \( \lambda \) to any grammar by adding a new start symbol \( S_0 \),

\[
S_0 \rightarrow S \mid \lambda
\]

**Theorem (Substitution)** Let \( G \) be a CFG. Suppose \( G \) contains

\[
A \rightarrow x_1 B x_2
\]

where \( A \) and \( B \) are different variables, and \( B \) has the productions

\[
B \rightarrow y_1 | y_2 | \ldots | y_n
\]

Then can construct \( G' \) from \( G \) by deleting

\[
A \rightarrow x_1 B x_2
\]

from \( P \) and adding to it

\[
A \rightarrow x_1 y_1 x_2 | x_1 y_2 x_2 | \ldots | x_1 y_n x_2
\]

Then, \( L(G) = L(G') \).

**Example:**

\[
S \rightarrow aBa
\]

becomes

\[
B \rightarrow aS \mid a
\]

**Definition:** A production of the form \( A \rightarrow Ax, A \in V, x \in (V \cup T)^* \) is left recursive.
**Example** Previous expression grammar was left recursive.

\[
\begin{align*}
E & \rightarrow E + T \mid T \\
T & \rightarrow T * F \mid F \\
F & \rightarrow I \mid (E) \\
I & \rightarrow a \mid b
\end{align*}
\]

A top-down parser would want to derive the leftmost terminal as soon as possible. But in the left recursive grammar above, in order to derive a sentential form that has the leftmost terminal, we have to derive a sentential form that has other terminals in it.

Derivation of \(a+b+a+a\) is:

\[
E \Rightarrow E + T \Rightarrow E + T + T \Rightarrow E + T + T + T \Rightarrow a + T + T + T
\]

We will eliminate the left recursion so that we can derive a sentential form with the leftmost terminal and no other terminals.

**Theorem** (Removing Left recursion) Let \(G=(V,T,S,P)\) be a CFG. Divide productions for variable \(A\) into left-recursive and non left-recursive productions:

\[
\begin{align*}
A & \rightarrow A_1 x_1 \mid A_2 x_2 \mid \ldots \mid A_n x_n \\
A & \rightarrow y_1 y_2 \ldots y_m
\end{align*}
\]

where \(x_i, y_i\) are in \((V \cup T)^*\).

Then \(G'=(V \cup \{Z\}, T, S, P')\) and \(P'\) replaces rules of form above by

\[
\begin{align*}
A & \rightarrow y_i z, i=1, 2, \ldots, m \\
Z & \rightarrow x_i z, i=1, 2, \ldots, n
\end{align*}
\]

**Example:**

\[
\begin{align*}
E & \rightarrow E + T | T \quad \text{becomes} \\
T & \rightarrow T * F | F \quad \text{becomes}
\end{align*}
\]

Now, Derivation of \(a+b+a+a\) is:
Useless productions

\[
\begin{align*}
S & \rightarrow aB \mid bA \\
A & \rightarrow aA \\
B & \rightarrow Sa \\
C & \rightarrow cBc \mid a \\
\end{align*}
\]

What can you say about this grammar?

**Theorem** (useless productions) Let G be a CFG. Then \( \exists G' \) that does not contain any useless variables or productions s.t. \( L(G) = L(G') \).

**To Remove Useless Productions:**

Let \( G = (V,T,S,P) \).

I. Compute \( V_1 = \{ \text{Variables that can derive strings of terminals} \} \)

1. \( V_1 = \emptyset \)
2. Repeat until no more variables added
   - For every \( A \in V \) with \( A \rightarrow x_1x_2 \ldots x_n, \ x_i \in (T^* \cup V_1) \), add \( A \) to \( V_1 \)
3. \( P_1 = \) all productions in \( P \) with symbols in \( (V_1 \cup T)^* \)

Then \( G_1 = (V_1,T,S,P_1) \) has no variables that can’t derive strings.

II. Draw Variable Dependency Graph

For \( A \rightarrow xBy \), draw \( A \rightarrow B \).

Remove productions for \( V \) if there is no path from \( S \) to \( V \) in the dependency graph. Resulting Grammar \( G' \) is s.t. \( L(G) = L(G') \) and \( G' \) has no useless productions.

**Example:**

\[
\begin{align*}
S & \rightarrow aB \mid bA \\
A & \rightarrow aA \\
B & \rightarrow Sa \mid b \\
C & \rightarrow cBc \mid a \\
D & \rightarrow bCb \\
E & \rightarrow Aa \mid b \\
\end{align*}
\]
**Theorem** (remove $\lambda$ productions) Let $G$ be a CFG with $\lambda$ not in $L(G)$. Then $\exists$ a CFG $G'$ having no $\lambda$-productions s.t. $L(G)=L(G')$.

**To Remove $\lambda$-productions**

1. Let $V_n = \{ A \mid \exists \text{ production } A \to \lambda \}$
2. Repeat until no more additions
   - if $B \to A_1 A_2 \ldots A_m$ and $A_i \in V_n$ for all $i$, then put $B$ in $V_n$
3. Construct $G'$ with productions $P'$ s.t.
   - if $A \to x_1 x_2 \ldots x_m \in P$, $m \geq 1$, then put all productions formed when $x_j$ is replaced by $\lambda$ (for all $x_j \in V_n$) s.t. $|\text{rhs}| \geq 1$ into $P'$.

**Example:**

\[
\begin{align*}
S & \to Ab \\
A & \to BCB \mid Aa \\
B & \to b \mid \lambda \\
C & \to cC \mid \lambda
\end{align*}
\]
**Definition** Unit Production

\[ A \rightarrow B \]

where \( A, B \in V \).

**Consider removing unit productions:**

Suppose we have

\[
\begin{align*}
A & \rightarrow B \\
B & \rightarrow a | ab
\end{align*}
\]

But what if we have

\[
\begin{align*}
A & \rightarrow B \\
B & \rightarrow C \\
C & \rightarrow A
\end{align*}
\]

**Theorem** (Remove unit productions) Let \( G=(V,T,S,P) \) be a CFG without \( \lambda \)-productions. Then \( \exists \) CFG \( G'=(V',T',S,P') \) that does not have any unit-productions and \( L(G)=L(G') \).

**To Remove Unit Productions:**

1. Find for each \( A \), all \( B \) s.t. \( A \Rightarrow B \) (Draw a dependency graph)
2. Construct \( G'=(V',T',S,P') \) by
   
   (a) Put all non-unit productions in \( P' \)
   
   (b) For all \( A \Rightarrow B \) s.t. \( B \rightarrow y_1 | y_2 | \ldots | y_n \in P' \), put \( A \rightarrow y_1 | y_2 | \ldots | y_n \in P' \)
Example:

S → AB
A → B
B → C | Bb
C → A | c | Da
D → A

**Theorem** Let L be a CFL that does not contain \( \lambda \). Then \( \exists \) a CFG for L that does not have any useless productions, \( \lambda \)-productions, or unit-productions.

**Proof**

1. Remove \( \lambda \)-productions
2. Remove unit-productions
3. Remove useless productions

Note order is very important. Removing \( \lambda \)-productions can create unit-productions! QED.
**Definition:** A CFG is in Chomsky Normal Form (CNF) if all productions are of the form

\[ A \rightarrow BC \quad \text{or} \quad A \rightarrow a \]

where \( A, B, C \in V \) and \( a \in T \).

**Theorem:** Any CFG \( G \) with \( \lambda \) not in \( L(G) \) has an equivalent grammar in CNF.

**Proof:**

1. Remove \( \lambda \)-productions, unit productions, and useless productions.
2. For every rhs of length > 1, replace each terminal \( x_i \) by a new variable \( C_j \) and add the production \( C_j \rightarrow x_i \).
3. Replace every rhs of length > 2 by a series of productions, each with rhs of length 2. QED.

**Example:**

\[
\begin{align*}
S & \rightarrow CBcd \\
B & \rightarrow b \\
C & \rightarrow Cc \mid e
\end{align*}
\]
**Definition:** A CFG is in Greibach normal form (GNF) if all productions have the form

\[ A \rightarrow ax \]

where \( a \in T \) and \( x \in V^* \)

**Theorem** For every CFG \( G \) with \( \lambda \) not in \( L(G) \), \( \exists \) a grammar in GNF.

**Proof:**

1. Rewrite grammar in CNF.
2. Relabel Variables \( A_1, A_2, \ldots A_n \)
3. Eliminate left recursion and use substitution to get all productions into the form:

\[
A_i \rightarrow A_jx_j, \ j > i \\
Z_i \rightarrow A_jx_j, \ j \leq n \\
A_i \rightarrow ax_i
\]

where \( a \in T \), \( x_i \in V^* \), and \( Z_i \) are new variables introduced for left recursion.

4. All productions with \( A_n \) are in the correct form, \( A_n \rightarrow ax_n \). Use these productions as substitutions to get \( A_{n-1} \) productions in the correct form. Repeat with \( A_{n-2}, A_{n-3} \), etc until all productions are in the correct form.