Computability A function $f$ with domain $D$ is *computable* if there exists some TM $M$ such that $M$ computes $f$ for all values in its domain.

Decidability A problem is *decidable* if there exists a TM that can answer yes or no to every statement in the domain of the problem.

The Halting Problem

Domain: set of all TMs and all strings $w$.

Question: Given coding of $M$ and $w$, does $M$ halt on $w$? (yes or no)

Theorem The halting problem is undecidable.

Proof: (by contradiction)

1. Assume there is a TM $H$ (or algorithm) that solves this problem.
   TM $H$ has 2 final states, $q_y$ represents yes and $q_n$ represents no.
   TM $H$ has input the coding of TM $M$ (denoted $w_M$) and input string $w$ and ends in state $q_y$ (yes) if $M$ halts on $w$ and ends in state $q_n$ (no) if $M$ doesn’t halt on $w$.

   $$H(w_M, w) = \begin{cases} 
   \text{(yes) halts in } q_y & \text{if } M \text{ halts on } w \\
   \text{(no) halts in } q_n & \text{if } M \text{ doesn’t halt on } w 
   \end{cases}$$

   TM $H$ always halts in a final state.

Construct TM $H'$ from $H$ such that $H'$ halts if $H$ ends in state $q_n$ and $H'$ doesn’t halt if $H$ ends in state $q_y$.

$$H'(w_M, w) = \begin{cases} 
   \text{halts} & \text{if } M \text{ doesn’t halt on } w \\
   \text{doesn’t halt} & \text{if } M \text{ halts on } w
   \end{cases}$$
Construct TM $\hat{H}$ from $H'$ such that $\hat{H}$ makes a copy of $w_M$ and then behaves like $H'$. (simulates TM $M$ on the input string that is the encoding of TM $M$, applies $M_w$ to $M_w$).

So $\hat{H}(w_M)$ runs $H'(w_M, w_M)$

$$\hat{H}(w_M) = \begin{cases} 
\text{halts} & \text{if } M \text{ doesn’t halt on } w_M \\
\text{doesn’t halt} & \text{if } M \text{ halts on } w_M
\end{cases}$$

Note that $\hat{H}$ is a TM.

There is some encoding of it, say $\hat{w}_{\hat{H}}$.

What happens if we run $\hat{H}$ with input $\hat{w}_{\hat{H}}$?

**Theorem** If the halting problem were decidable, then every recursively enumerable language would be recursive. Thus, the halting problem is undecidable.

- **Proof**: Let $L$ be an RE language over $\Sigma$.
  
  Let $M$ be the TM such that $L = L(M)$.
  
  Let $H$ be the TM that solves the halting problem.
A problem A is reduced to problem B if the decidability of B follows from the decidability of A. Then if we know B is undecidable, then A must be undecidable.

**State-entry problem** Given TM $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$, state $q \in Q$, and string $w \in \Sigma^*$, is state $q$ ever entered when $M$ is applied to $w$?

This is an undecidable problem!

- **Proof:** We will reduce this problem to the halting problem.

  Suppose we have a TM $E$ to solve the state-entry problem.
  TM $E$ takes as input the coding of a TM $M$ (denoted by $w_M$), a string $w$ and a state $q$. TM $E$ answers *yes* if state $q$ is entered and *no* if state $q$ is not entered.

  Construct TM $E'$ which does the following. On input $w_M$ and $w$ $E'$ first examines the transition functions of $M$. Whenever $\delta$ is not defined for some state $q_i$ and symbol $a$ add the transition $\delta(q_i, a) = (q, a, R)$. Let this new state $q$ be the only final state. Let $M'$ be the modified TM. Next, simulate TM $E$ on input $w_M'$, $w$ and $q$.

  $$E'(w_M, w) = \begin{cases} 
  M \text{ halts on } w & \text{if } M' \text{ enters state } q \\
  M \text{ doesn't halt on } w & \text{if } M' \text{ doesn't enter state } q 
  \end{cases}$$

  TM $E'$ determines if $M$ halts on $w$. If $M$ halts on $w$ then TM $E'$ will enter state $q$ in $M'$ and answer *yes*. If $M$ doesn’t halt on $w$ then TM $E'$ will not enter state $q$, so it will answer *no*. Since the state-entry problem is decidable, $E$ always gives an answer yes or no.

  But the halting problem is undecidable. Contradiction! Thus, the state-entry problem must be undecidable. QED.

There are some more examples of undecidability in section 12.1.