Section: Decidability

Computability A function $f$ with domain $D$ is computable if there exists some TM $M$ such that $M$ computes $f$ for all values in its domain.

Decidability A problem is decidable if there exists a TM that can answer yes or no to every statement in the domain of the problem.
The Halting Problem

Domain: set of all TMs and all strings $w$.

Question: Given coding of $M$ and $w$, does $M$ halt on $w$?
Theorem The halting problem is undecidable.

Proof: (by contradiction)

Assume there is a TM H (or algorithm) that solves this problem. TM H has 2 final states, $q_y$ represents yes and $q_n$ represents no.

$$H(w_M, w) = \begin{cases} 
\text{halts } q_y \text{ if } M \text{ halts on } w \\
\text{halts } q_n \text{ if } M \text{ doesn't halt on } w 
\end{cases}$$

TM H always halts in a final state.
Construct TM $H'$ from $H$

$$H'(w_M, w) = \begin{cases} 
\text{halts} & \text{if } M \text{ not halt on } w \\
\text{not halt} & \text{if } M \text{ halts on } w 
\end{cases}$$

Construct TM $\hat{H}$ from $H'$

$$\hat{H}(w_M) = \begin{cases} 
\text{halts} & \text{if } M \text{ not halt on } w_M \\
\text{not halt} & \text{if } M \text{ halts on } w_M 
\end{cases}$$

Note that $\hat{H}$ is a TM.
There is some encoding of it, say $\hat{w}_{\hat{H}}$.
What happens if we run $\hat{H}$ with input $\hat{w}_{\hat{H}}$?
Theorem If the halting problem were decidable, then every recursively enumerable language would be recursive. Thus, the halting problem is undecidable.

• Proof: Let \( L \) be an RE language over \( \Sigma \).
  
  Let \( M \) be the TM such that \( L = L(M) \).
  
  Let \( H \) be the TM that solves the halting problem.
A problem A is *reduced* to problem B if the decidability of B follows from the decidability of A. Then if we know B is undecidable, then A must be undecidable.
State-entry problem Given TM
\( M = (Q, \Sigma, \Gamma, \delta, q_0, B, F) \), state \( q \in Q \), and
string \( w \in \Sigma^* \), is state \( q \) ever entered
when \( M \) is applied to \( w \)?

This is an undecidable problem!

- Proof:

  TM \( E \) solves state-entry problem

\[
E'(w_M, w) = \begin{cases} 
M \text{ halts on } w & \text{ if }? \\
M \text{ doesn't halt on } w & \text{ if }?
\end{cases}
\]