CPS 590.4
Brief introduction to linear and mixed integer programming

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Linear programs: example

- We make reproductions of two paintings

  \[
  \text{maximize } 3x + 2y
  \]

\text{subject to}

\[
4x + 2y \leq 16
\]

\[
x + 2y \leq 8
\]

\[
x + y \leq 5
\]

\[
x \geq 0
\]

\[
y \geq 0
\]

- Painting 1 sells for $30, painting 2 sells for $20
- Painting 1 requires 4 units of blue, 1 green, 1 red
- Painting 2 requires 2 blue, 2 green, 1 red
- We have 16 units blue, 8 green, 5 red
Solving the linear program graphically

**maximize** $3x + 2y$

**subject to**

$4x + 2y \leq 16$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

**optimal solution:** $x=3$, $y=2$
Modified LP

maximize $3x + 2y$

subject to

$4x + 2y \leq 15$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

Optimal solution: $x = 2.5$, $y = 2.5$

Solution value = $7.5 + 5 = 12.5$

Half paintings?
Integer (linear) program

Maximize $3x + 2y$

Subject to

$4x + 2y \leq 15$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$, integer

$y \geq 0$, integer

Optimal IP solution: $x=2$, $y=3$ (objective 12)

Optimal LP solution: $x=2.5$, $y=2.5$ (objective 12.5)
Mixed integer (linear) program

maximize $3x + 2y$

subject to

$4x + 2y \leq 15$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$, integer

optimal IP solution: $x=2$, $y=3$ (objective 12)

optimal LP solution: $x=2.5$, $y=2.5$ (objective 12.5)

optimal MIP solution: $x=2.75$, $y=2$ (objective 12.25)
Solving linear/integer programs

• Linear programs can be solved efficiently
  – Simplex, ellipsoid, interior point methods…

• (Mixed) integer programs are NP-hard to solve
  – Quite easy to model many standard NP-complete problems as integer programs (try it!)
  – Search type algorithms such as branch and bound

• Standard packages for solving these
  – GNU Linear Programming Kit, CPLEX, …

• LP relaxation of (M)IP: remove integrality constraints
  – Gives upper bound on MIP (~admissible heuristic)
Exercise in modeling: knapsack-type problem

• We arrive in a room full of precious objects
• Can carry only 30kg out of the room
• Can carry only 20 liters out of the room
• Want to maximize our total value
• Unit of object A: 16kg, 3 liters, sells for $11
  – There are 3 units available
• Unit of object B: 4kg, 4 liters, sells for $4
  – There are 4 units available
• Unit of object C: 6kg, 3 liters, sells for $9
  – Only 1 unit available
• What should we take?
Exercise in modeling: cell phones (set cover)

• We want to have a working phone in every continent (besides Antarctica)
• … but we want to have as few phones as possible

• Phone A works in NA, SA, Af
• Phone B works in E, Af, As
• Phone C works in NA, Au, E
• Phone D works in SA, As, E
• Phone E works in Af, As, Au
• Phone F works in NA, E
Exercise in modeling: hot-dog stands

- We have two hot-dog stands to be placed in somewhere along the beach
- We know where the people that like hot dogs are, how far they are willing to walk
- Where do we put our stands to maximize #hot dogs sold? (price is fixed)

<table>
<thead>
<tr>
<th>Location</th>
<th>#Customers</th>
<th>Willing to Walk</th>
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<tr>
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<td>4</td>
</tr>
<tr>
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</tr>
<tr>
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<tr>
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