Splines

Cubic B-Splines
Nonuniform Rational B-Splines
Rendering by Subdivision
Curves and Surfaces in OpenGL
[Angel, Ch 10.7-10.14]

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Frank Pfenning
Carnegie Mellon University

http://www.cs.cmu.edu/~fp/courses/graphics/
Review

• Cubic polynomial form for curve

\[ p(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3 = \sum_{k=0}^{3} c_k u^k \]

• Each \( c_k \) is a column vector \([c_{kx} \ c_{ky} \ c_{kz}]^T\)

• Solve for \( c_k \) given control points

• Interpolation: 4 points

• Hermite curves: 2 endpoints, 2 tangents

• Bezier curves: 2 endpoints, 2 tangent points
Splines

- Approximating more control points

- C\(^0\) continuity: points match
- C\(^1\) continuity: tangents (derivatives) match
- C\(^2\) continuity: curvature matches
- With Bezier segments or patches: C\(^0\)
B-Splines

- Use 4 points, but approximate only middle two

- Draw curve with overlapping segments 0-1-2-3, 1-2-3-4, 2-3-4-5, 3-4-5-6, etc.
- Curve may miss all control points
- Smoother at joint points
Cubic B-Splines

- Need m+2 control points for m cubic segments
- Computationally 3 times more expensive
- $C^2$ continuous at each interior point
- Derive as follows:
  - Consider two overlapping segments
  - Enforce $C^0$ and $C^1$ continuity
  - Employ symmetry
  - $C^2$ continuity follows
Deriving B-Splines

• Consider points
  – \( p_{i-2}, p_{i-1}, p_i, p_{i+1} \)
  – \( p(0) \approx p_{i-1}, p(1) \approx p_i \)
  – \( p_{i-3}, p_{i-2}, p_{i-1}, p_i \)
  – \( q(0) \approx p_{i-2}, q(1) \approx p_{i-1} \)

• **Condition 1:** \( p(0) = q(1) \)
  – Symmetry: \( p(0) = q(1) = \frac{1}{6}(p_{i-2} + 4p_{i-1} + p_i) \)

• **Condition 2:** \( p'(0) = q'(1) \)
  – Geometry: \( p'(0) = q'(1) = \frac{1}{2} \left( (p_i - p_{i-1}) + (p_{i-1} - p_{i-2}) \right) \)
  = \( \frac{1}{2} (p_i - p_{i-2}) \)
B-Spline Geometry Matrix

- Conditions at $u = 0$
  - $p(0) = c_0 = \frac{1}{6} (p_{i-2} + 4p_{i-1} + p_i)$
  - $p'(0) = c_1 = \frac{1}{2} (p_i - p_{i-2})$

- Conditions at $u = 1$
  - $p(1) = c_0 + c_1 + c_2 + c_3 = \frac{1}{6} (p_{i-1} + 4p_i + p_{i+1})$
  - $p'(1) = c_1 + 2c_2 + 3c_3 = \frac{1}{2} (p_{i+1} - p_{i-1})$

\[
\begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3 \\
\end{bmatrix} = M_S \begin{bmatrix}
p_{i-2} \\
p_{i-1} \\
p_i \\
p_{i+1} \\
\end{bmatrix}, \quad M_S = \frac{1}{6} \begin{bmatrix}
1 & 4 & 1 & 0 \\
-3 & 0 & 3 & 0 \\
3 & -6 & 3 & 0 \\
-1 & 3 & -3 & 1 \\
\end{bmatrix}
\]
Blending Functions

• Calculate cubic blending polynomials

\[ b(u) = M_S^T u = \frac{1}{6} \begin{pmatrix} (1 - u)^3 \\ 4 - 6u^2 + 3u^3 \\ 1 + 3u + 3u^2 - 3u^3 \\ u^3 \end{pmatrix} \]

• Note symmetries
Convex Hull

• For $0 \leq u \leq 1$, have $0 \leq b_k(u) \leq 1$
• Recall:
  \[ p(u) = b_{i-2}(u)p_{i-2} + b_{i-1}(u)p_{i-1} + b_i(u)p_i + b_{i+1}(u)p_{i+1} \]
• So each point $p(u)$ lies in convex hull of $p_k$
Spline Basis Functions

- Total contribution $B_i(u)p_i$ of $p_i$ is given by

$$B_i(u) = \begin{cases} 
0 & u < i - 2 \\
b_0(u + 2) & i - 2 \leq u < i - 1 \\
b_1(u + 1) & i - 1 \leq u \leq i \\
b_2(u) & i \leq u < i + 1 \\
b_3(u - 1) & i + 1 \leq u < i + 2 \\
0 & i - 2 \leq u 
\end{cases}$$
Spline Surface

- As for Bezier patches, use 16 control points
- Start with blending functions

\[ p(u, v) = \sum_{i=0}^{3} \sum_{k=0}^{3} b_i(u)b_k(v)p_{ik} \]

- Need 9 times as many splines as for Bezier
Assessment: Cubic B-Splines

- More expensive than Bezier curves or patches
- Smoother at join points
- Local control
  - How far away does a point change propagate?
- Contained in convex hull of control points
- Preserved under affine transformation
- How to deal with endpoints?
  - Closed curves (uniform periodic B-splines)
  - Non-uniform B-Splines (multiplicities of knots)
General B-Splines

• Generalize from cubic to arbitrary order
• Generalize to different basis functions
• Read: [Angel, Ch 10.8]
• Knot sequence $u_{\text{min}} = u_0 \leq \ldots \leq u_n = u_{\text{max}}$
• Repeated points have higher “gravity”
• Multiplicity 4 means point must be interpolated
• $\{0, 0, 0, 0, 1, 2, \ldots, n-1, n, n, n, n\}$ solves boundary problem
Nonuniform Rational B-Splines (NURBS)

• Exploit homogeneous coordinates

\[
p_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \simeq \begin{bmatrix} w_i \\ x_i \\ y_i \\ z_i \\ 1 \end{bmatrix} = q_i
\]

• Use perspective division to renormalize

\[
p(u) = \frac{\sum_{i=0}^{n} B_i(u) w_i p_i}{\sum_{i=0}^{n} B_i(u) w_i}
\]

• Each component of \( p(u) \) is rational function of \( u \)
• Points not necessarily uniformly (NURBS)
NURBS Assessment

• Convex-hull and continuity props. of B-splines
• Preserved under *perspective* transformations
  – Curve with transformed points = transformed curve
• Widely used (including OpenGL)
Outline

- Cubic B-Splines
- Nonuniform Rational B-Splines (NURBS)
- Rendering by Subdivision
- Curves and Surfaces in OpenGL
Rendering by Subdivision

• Divide the curve into smaller subpieces
• Stop when “flat” or at fixed depth
• How do we calculate the sub-curves?
  – Bezier curves and surfaces: easy (next)
  – Other curves: convert to Bezier!
Subdividing Bezier Curves

- Given Bezier curve by $p_0$, $p_1$, $p_2$, $p_3$
- Find $l_0$, $l_1$, $l_2$, $l_3$ and $r_0$, $r_1$, $r_2$, $r_3$
- Subcurves should stay the same!
Construction of Bezier Subdivision

- Use algebraic reasoning

- \( l(0) = l_0 = p_0 \)
- \( l(1) = l_3 = p(1/2) = 1/8(p_0 + 3p_1 + 3p_2 + p_3) \)
- \( l'(0) = 3(l_1 - l_0) = p'(0) = 3/2 (p_1 - p_0) \)
- \( l'(1) = 3(l_3 - l_2) = p'(1/2) = 3/8(-p_0 - p_1 + p_2 + p_3) \)
- Note parameter substitution \( v = 2u \) so \( dv = 2du \)
**Geometric Bezier Subdivision**

- Can also calculate geometrically

\[ \begin{align*}
    l_1 &= \frac{1}{2}(p_0 + p_1), \quad r_2 = \frac{1}{2} (p_2 + p_3) \\
    l_2 &= \frac{1}{2} (l_1 + \frac{1}{2} (p_1 + p_2)), \quad r_1 = \frac{1}{2} (r_2 + \frac{1}{2}(p_1 + p_2)) \\
    l_3 &= r_0 = \frac{1}{2} (l_2 + r_1), \quad l_0 = p_0, \quad r_3 = p_3
\end{align*} \]
Recall: Bezier Curves

- Recall

\[ u^T = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \]

- Express

\[ p(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3 \]

\[ = u^T \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = u^T M_B \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \]

\[ M_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & -1 \end{bmatrix} \]
Subdividing Other Curves

- Calculations more complex
- Trick: transform control points to obtain identical curve as Bezier curve!
- Then subdivide the resulting Bezier curve
- Bezier: \( p(u) = u^T M_b \ p \)
- Other curve: \( p(u) = u^T M \ q, \ M \) geometry matrix
- Solve: \( q = M^{-1} M_b \ p \) with \( p = M_b^{-1} M \ q \)
Example Conversion

- From cubic B-splines to Bezier:

\[ \mathbf{M}_{B}^{-1} \mathbf{M}_{S} = \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 1 \end{bmatrix} \]

- Calculate Bezier points \( p \) from \( q \)
- Subdivide as Bezier curve
Subdivision of Bezier Surfaces

- Slightly more complicated
- Need to calculate interior point
- Cracks may show with uneven subdivision
- See [Angel, Ch 10.9.4]
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Curves and Surface in OpenGL

- Central mechanism is **evaluator**
- Defined by array of control points
- Evaluate coordinates at u (or u and v) to generate vertex
- Define Bezier curve: \( type = \text{GL\_MAP\_VERTEX\_3} \)
  \[
  \text{glMap1f}(type, u_0, u_1, \text{stride}, \text{order}, \text{point\_array})
  \]
- Enable evaluator
  \[
  \text{glEnable}(type)
  \]
- Evaluate Bezier curve
  \[
  \text{glEvalCoord1f}(u)
  \]
Example: Drawing a Bezier Curve

- 4 control points

```c
GLfloat ctrlpoints[4][3] = {
    {-4.0, -4.0, 0.0}, {-2.0, 4.0, 0.0},
    {2.0, -4.0, 0.0}, {4.0, 4.0, 0.0}};
```

- Initialize

```c
void init()
{
    ...;
    glMap1f(GL_MAP1_VERTEX_3, 0.0, 1.0, 3, 4, &ctrlpoints[0][0]);
    glEnable(GL_MAP1_VERTEX_3);
}
```
Evaluating Coordinates

• Use a fixed number of points, num_points

void display()
{
    glBegin(GL_LINE_STRIP);
    for (i = 0; i <= num_points; i++)
        glEvalCoord1f((GLfloat)i/(GLfloat)num_points);
    glEnd();
    ...
}
Drawing the Control Points

• To illustrate Bezier curve

```c
void display()
{
    ...  
    glPointSize(5.0);
    glColor3f(1.0, 1.0, 0.0);
    glBegin(GL_POINTS);
    for (i = 0; i < 4; i++)
        glVertex3fv(&ctrlpoints[i][0]);
    glEnd();
    glFlush();
}
```
Resulting Images

n = 5  
n = 20
Bezier Surfaces

• Create evaluator in two parameters u and v

```c
glMap2f(GL_MAP2_VERTEX_3,
    u0, u1, ustride, uorder,
    v0, v1, vstride, vorder, point_array);
```

• Enable, also automatic calculation of normal

```c
glEnable(GL_MAP2_VERTEX_3);
glEnable(GL_AUTO_NORMAL);
```

• Evaluate at parameters u and v

```c
glEvalCoord2f(u, v);
```
Grids

- Convenience for uniform evaluators
- Define grid \( (nu = \text{number of } u \text{ division}) \)
  
  \[
  \text{glMapGrid2f(nu, } u_0, u_1, nv, v_0, v_1)\];

- Evaluate grid

  \[
  \text{glEvalMesh2(mode, } i_0, i_1, k_0, k_1)\];

- \textit{mode} = \text{GL\_POINT, GL\_LINE, or GL\_FILL}

- \(i\) and \(k\) define subrange
Example: Bezier Surface Patch

• Use 16 control points

  GLfloat ctrlpoints[4][4][3] = {...};

• Initialize 2-dimensional evaluator

  void init(void)
  { ...
    glMap2f(GL_MAP2_VERTEX_3, 0, 1, 3, 4,
           0, 1, 12, 4, &ctrlpoints[0][0][0]);
    glEnable(GL_MAP2_VERTEX_3);
    glEnable(GL_AUTO_NORMAL);
    glMapGrid2f(20, 0.0, 1.0, 20, 0.0, 1.0);
  }
Evaluating the Grid

- Use full range

```c
void display(void)
{
    ... 
    glPushMatrix();
    glRotatef(85.0, 1.0, 1.0, 1.0);
    glEvalMesh2(GL_FILL, 0, 20, 0, 20);
    glPopMatrix();
    glFlush();
}
```
NURBS Functions

• Higher-level interface
• Implemented in GLU using evaluators
• Create a NURBS renderer

```c
theNurb = gluNewNurbsRenderer();
```

• Set NURBS properties

```c
gluNurbsProperty(theNurb, GLU_DISPLAY_MODE, GLU_FILL);
gluNurbsCallback(theNurb, GLU_ERROR, nurbsError);
```
Displaying NURBS Surfaces

• Specify knot arrays for splines

```c
GLfloat knots[8] = {0.0, 0.0, 0.0, 0.0, 1.0, 1.0, 1.0, 1.0};
gluBeginSurface(theNurb);
gluNurbsSurface(theNurb,
               8, knots, 8, knots,
               4 * 3, 3, &ctlpoints[0][0][0],
               4, 4, GL_MAP2_VERTEX_3);
gluEndSurface(theNurb);
```

• For more see [Red Book, Ch. 12]
Summary

• Cubic B-Splines
• Nonuniform Rational B-Splines (NURBS)
• Rendering by Subdivision
• Curves and Surfaces in OpenGL
Reminders

• Assignment 3 due tonight
• Assignment 4 out some time today
• Midterm will cover curves and surfaces
• Next Tuesday: Textures?
• Next Thursday: John Ketchpaw