

Lecture 7

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1 Overview

In the last lecture, we talked about Karger’s random contraction algorithm for constructing a global min-cut, and a simple uniform sampling algorithm for constructing a sparse subgraph that approximately preserves all cuts of the original graph (also due to Karger). We observed that the sampling algorithm would not help when the global min-cut size is too small, and proposed to sample each edge with probability proportional to the “connectivity” of that edge, instead of a uniform probability dependent on the global min-cut size. In this lecture, we introduce four connectivity parameters, explain the intuition behind each parameter, and show the relationship between them.

2 Connectivity Parameters

In this section, we define four connectivity parameters for an edge in a graph, namely, connectivity, strength, conductance, and Nagamochi-Ibaraki (NI) index. Let $G = (V, E)$ be an undirected graph. We always use n to denote $|V|$ and m to denote $|E|$.

Definition 1. (Connectivity) *The connectivity of an edge $e \in E$ of a graph $G = (V, E)$, denoted as λ_e , is the size of the smallest cut of G that contains edge e .*

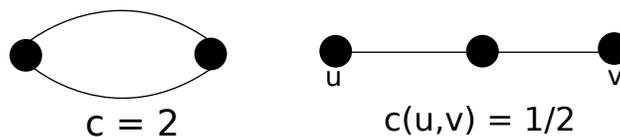
Let the two endpoints of e be $u, v \in V$. In other words, λ_e is the minimum number of edges to remove to disconnect u from v . For example, every edge in a tree has connectivity 1 and every edge in a complete graph has connectivity $n - 1$. For the graph in Figure 1(a), we have $\lambda_{e_1} = 2$ and $\lambda_{e_2} = 5$.

Definition 2. (Strength) *The strength of an edge $e \in E$ of a graph $G = (V, E)$, denoted as s_e , is the maximum positive integer k such that e belongs to a k -connected induced subgraph of G . Here, a graph is said to be k -connected if the min-cut of the graph is at least k .*

For the graph in Figure 1(a), we have $s_{e_1} = 2$ because (i) the whole graph contains e_1 and is 2-connected, and (ii) one vertex of e_1 has degree 2, so any subgraph containing e_1 has min-cut at most 2. Also, we have $s_{e_2} = 2$ with a similar argument.

Definition 3. (Conductance) *The conductance of an edge $e \in E$ of a graph $G = (V, E)$, denoted as c_e , is the electrical conductance between the endpoints of e if every edge in E is replaced by a resistor of unit resistance.*

Recall that the conductance is the inverse of resistance. Kirchhoff’s law says that the resistances are summed in series, and the conductances are summed in parallel. See the following for an illustration.



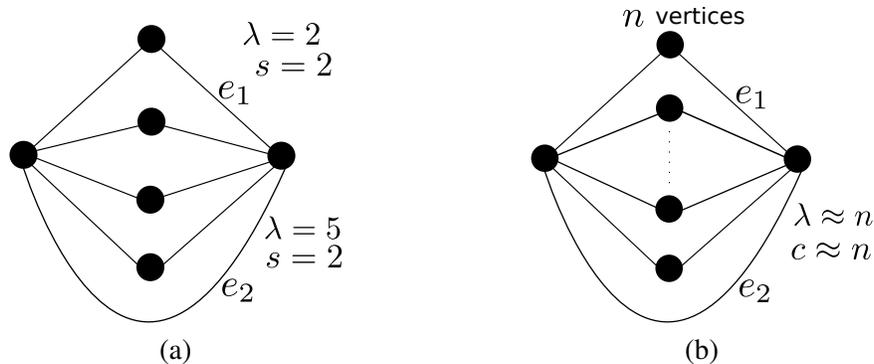
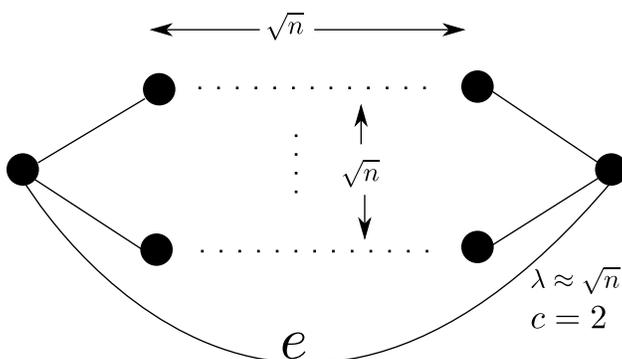


Figure 1: Demonstrating examples.

For the graph in Figure 1(b), both the connectivity and conductance of edge e_2 are roughly n . However, connectivity and conductance can also be very different, such as for edge e of the graph below.

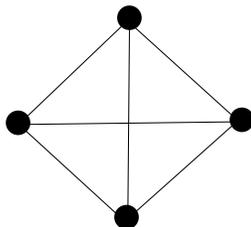


Definition 4. (Nagamochi-Ibaraki) The Nagamochi-Ibaraki index of an edge $e \in E$ of a graph $G = (V, E)$, denoted as t_e , is index of the tree containing e in any greedy spanning forest packing of G .

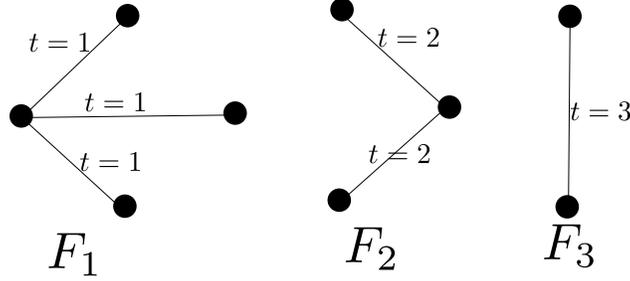
A spanning forest packing of a graph G can be obtained by iteratively performing the following steps until no edge is left:

- (i) find a spanning forest among the remaining edges
- (ii) remove the edges found

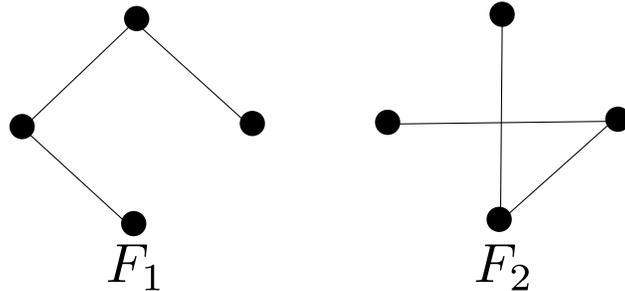
For example, for the complete graph with 4 vertices K_4 below,



one spanning forest packing is as follows (vertices with no edges attached are not drawn). So edges in F_i have NI index i for $i = 1, 2, 3$.



Note that a key difference between NI index and the other three connectivity parameters is that the NI index is not unique, because the spanning forest packing of G is not unique. For example, below is another spanning forest packing of K_4 .



3 Properties of Connectivity Parameters

It turns out that, for any edge e , its connectivity is always no less than its strength, conductance, or NI index.

Lemma 1. For any edge $e \in E$, $\lambda_e \geq s_e$.

Proof. Consider the min-cut containing edge e , denoted as (S, \bar{S}) . Then λ_e is equal to the size of the cut (S, \bar{S}) . Let G_e denote any subgraph containing e . Then G_e must contain vertices from both S and \bar{S} because the edge e itself is between S and \bar{S} . Thus, the size of the cut $(G_e \cap S, G_e \cap \bar{S})$ for G_e is at most the size of cut (S, \bar{S}) for the original graph G . Therefore, the connectivity of G_e is at most λ_e , and $s_e \leq \lambda_e$. \square

Lemma 2. For any edge $e \in E$, $\lambda_e \geq t_e$.

Proof. Suppose the NI index of e is $t_e = k$, i.e., e appears in the k -th spanning forest. That means in all the previous $k - 1$ spanning forests, the endpoints of e are connected. So there are k disjoint paths connecting the endpoints of e , which means that the min-cut containing e must contain at least k edges. \square

Lemma 3. For any edge $e \in E$, $\lambda_e \geq c_e$.

Proof. The proof relies on the *Raleigh's Monotonicity Principle*: if resistances are not increased in a network, then the effective resistance between any two vertices does not increase either.

Consider the min-cut (S, \bar{S}) containing e . Set the resistance of an edge to 0 if it is inside S or \bar{S} , and 1 if it is between S and \bar{S} (i.e., in the min-cut). It is easy to see that the conductance of this resistance setting c'_e is equal to the size of the min-cut (S, \bar{S}) , i.e., λ_e . The original setting of resistance (1 for every edge) is an increment over this setting, and thus the effective conductance c_e with the original setting of resistance, is at most λ_e . \square

Finally, we show some properties of the connectivity parameters.

Lemma 4. *The following four inequalities holds:*

$$(i) \sum_{e \in E} \frac{1}{\lambda_e} \leq n - 1$$

$$(ii) \sum_{e \in E} \frac{1}{s_e} \leq n - 1$$

$$(iii) \sum_{e \in E} r_e = \sum_{e \in E} \frac{1}{c_e} = n - 1$$

$$(iv) \sum_{e \in E} \frac{1}{t_e} = (n - 1)H_n = O(n \log n).$$

Proof. We first prove (iv). Consider any greedy spanning forest packing of G , denoted as F_1, F_2, \dots, F_k . There are at most $n - 1$ edges in each F_i , so

$$\begin{aligned} \sum_{e \in E} \frac{1}{t_e} &= \sum_{i=1}^k \sum_{e \in F_i} \frac{1}{t_e} \\ &= \sum_{i=1}^k \sum_{e \in F_i} \frac{1}{i} \\ &\leq \sum_{i=1}^k (n - 1) \frac{1}{i} \\ &= (n - 1)H_k. \end{aligned}$$

But, $k \leq n - 1$ because any vertex v is adjacent to at most $n - 1$ edges, and each spanning forest consumes one edge adjacent to v if there is any such edge remaining.

Next, we prove (iii). The proof relies on a theorem: the effective resistance of any edge e is equal to the probability of e being in a uniformly random spanning tree. Draw a random spanning tree T . Let $x_e \in \{0, 1\}$ be an indicator random variable such that $x_e = 1$ iff e is in T . So the expected number of edges in T is $\sum_{e \in E} \Pr(x_e = 1) = \sum_{e \in E} r_e$. But every spanning tree has $n - 1$ edges. Thus (iii) is true.

Finally, we prove (ii). We prove this by induction. The base case contains a single vertex and is obviously true. Suppose (ii) is true for all graphs with at most $n - 1$ vertices. Consider a graph G with n vertices. Let (S, \bar{S}) be the global min-cut of size λ . Then it is easy to show that $s_e = \lambda$ for edges in the global min-cut. Therefore,

$$\begin{aligned} \sum_{e \in E} \frac{1}{s_e} &= \sum_{e \in G_S} \frac{1}{s_e} + \sum_{e \in G_{\bar{S}}} \frac{1}{s_e} + \sum_{e \in (S, \bar{S})} \frac{1}{s_e} \\ &\leq (|S| - 1) + (|\bar{S}| - 1) + \lambda \frac{1}{\lambda} \\ &= n - 1 \end{aligned}$$

(i) is automatically true because $\lambda_e \geq s_e$ for all e . □