# Uncertainty

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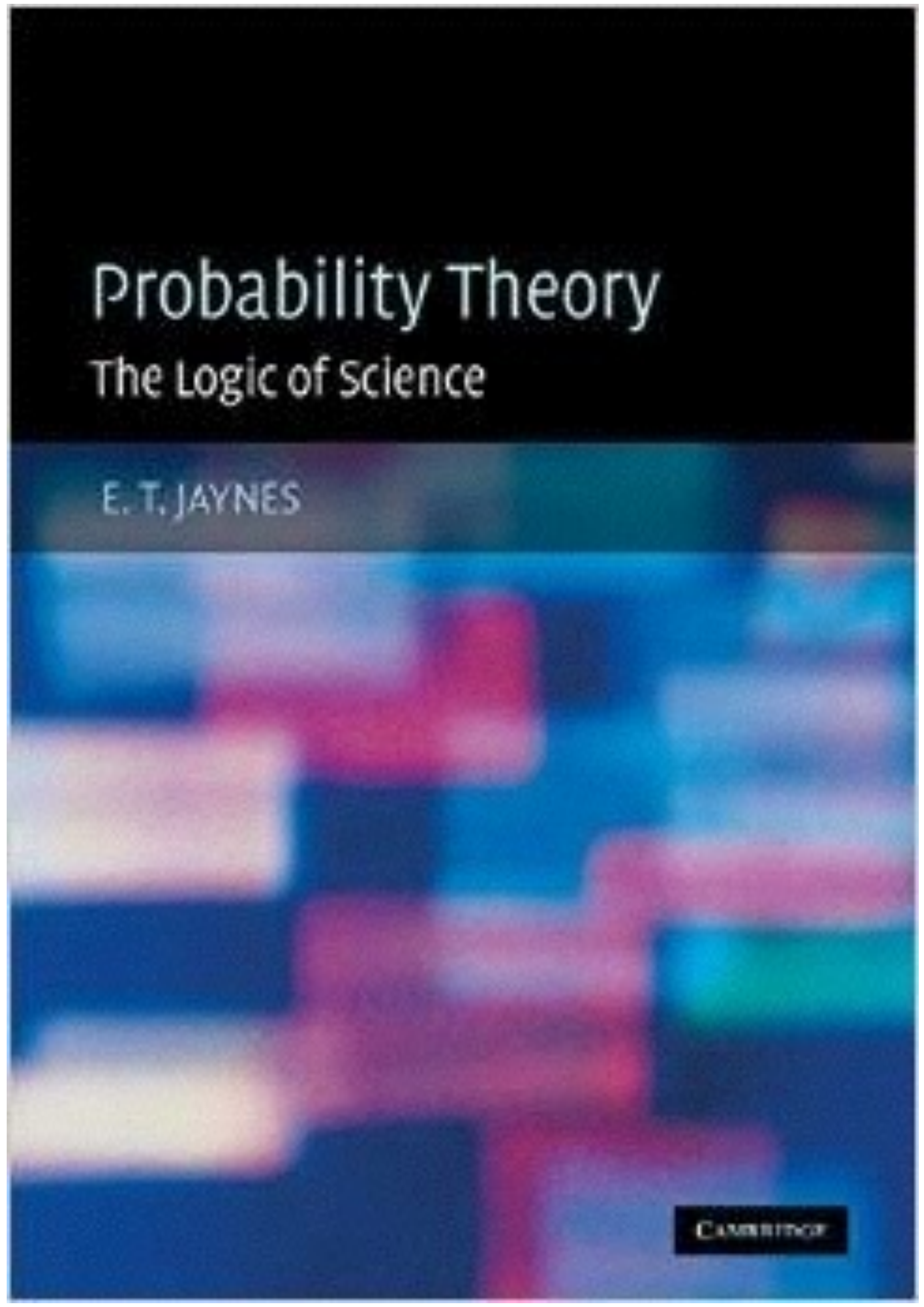
# Logic is Insufficient



The world is not deterministic.
There is such thing as a fact.
Generalization is hard.
Sensors and actuators are noisy.
Plans fail.
Models are not perfect.
Learned models are *especially* imperfect.

$$\forall x, Fruit(x) \implies Tasty(x)$$


Probability Theory
The Logic of Science
E. T. JAYNES

# Probabilities

Powerful tool for reasoning about uncertainty.

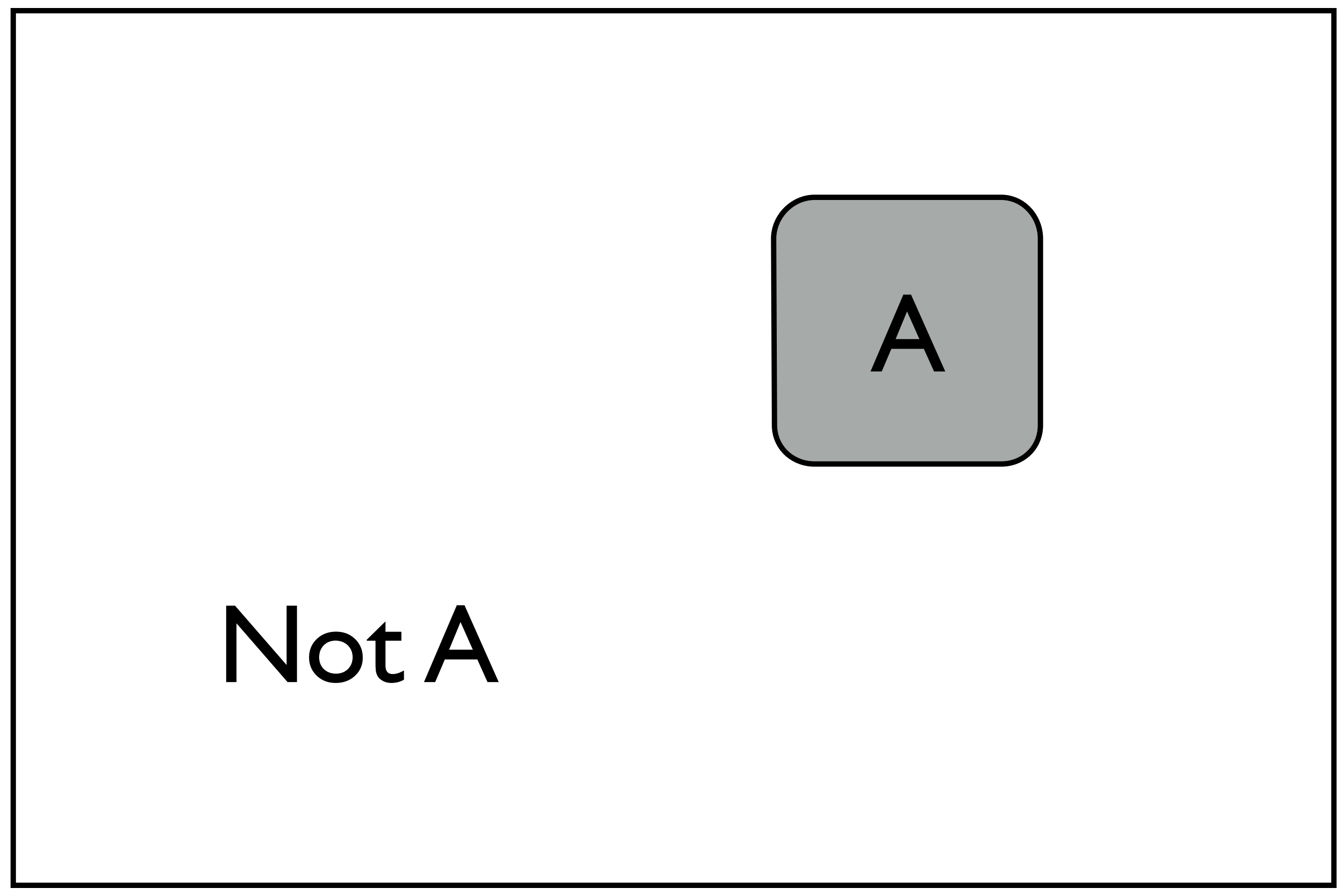
But, they're tricky:

- Intuition often wrong or inconsistent.
- Difficult to *get*.

What do probabilities **really mean**?

# Relative Frequencies

Defined over *events*.




P(A): probability random event falls in *A*, rather than *Not A*.
Works well for dice and coin flips!

# Relative Frequencies

But this feels limiting.

*What is the probability that the Blue Devils will beat Virginia on Saturday?*

- Meaningful question to ask.
- Can't count frequencies (except naively).
- Only really happens once.

In general, *all events only happen once*.

# Probabilities and Beliefs



Suppose I flip a coin and hide outcome.

- What is P(Heads)?

This is a statement about *a belief*, not *the world*.
(the world is in exactly one state, with prob. 1)

Assigning truth values to probabilities is tricky - must reference speaker's *state of knowledge*.

**Frequentists**: probabilities come from relative frequencies.
**Subjectivists**: probabilities are degrees of belief.

# For Our Purposes

No two events are identical, or completely unique.

Use probabilities as beliefs, but allow data (relative frequencies) to influence these beliefs.

**We use *Bayes' Rule* to combine prior beliefs with new data.**

**Can prove that a person who holds a system of beliefs inconsistent with probability theory can be fooled.**

# To The Math

Probabilities talk about *random variables:*

- X, Y, Z, with domains d(X), d(Y), d(Z).
- Domains may be *discrete* or *continuous*.
- X = x: RV X has taken value x.
- P(x) is short for P(X = x).

# Examples

X: RV indicating winner of Duke vs. Virginia game.

$$d(X) = \{\text{Duke}, \text{Virginia}, \text{tie}\}.$$

A probability is associated with each *event* in the domain:

- $P(X = \text{Duke}) = 0.8$
- $P(X = \text{Virginia}) = 0.19$
- $P(X = \text{tie}) = 0.01$

Note: probabilities over *the entire event space* must sum to 1.

# Expectation

Common use of probabilities: each event has *numerical value*.

Example: 6 sided die.
What is the *average die value?*

$$\frac{(1+2+3+4+5+6)}{6} = 3.5$$

In general, given RV X and function *f(x)*:

$$E[f(x)] = \sum_{x} P(x) f(x)$$

# Expectation

For example, in min-max search, we assumed the opposing player took the min valued action (for us).

But that assumes perfect play. If we have *a probability distribution over the player's actions*, we can calculate their *expected value* (as opposed to min value) for each action.

Result: *expecti-max* algorithm.

# Kolmogorov's Axioms of Probability

- 0 <= P(x) <= 1
- P(true) = 1, P(false) = 0
- P(a or b) = P(a) + P(b) - P(a and b)

Sufficient to completely specify probability theory for discrete variables.

# Multiple Events

When several variables are involve, think about *atomic events*.

- Complete assignment of all variables.
- All possible events.
- Mutually exclusive.

RVs: Raining, Cold (both binary):

| **Raining** | **Cold** | **Prob.** |
|---|---|---|
| True | True | 0.3 |
| True | False | 0.1 |
| False | True | 0.4 |
| False | False | 0.2 |

joint distribution

Note: *still adds up to 1.*

# Joint Probability Distribution

Probabilities to all possible atomic events *(grows fast)*

| **Raining** | **Cold** | **Prob.** |
|---|---|---|
| True | True | 0.3 |
| True | False | 0.1 |
| False | True | 0.4 |
| False | False | 0.2 |

Can define individual probabilities in terms of JPD:

*P(Raining) = P(Raining, Cold) + P(Raining, not Cold) = 0.4.*

$$P(a) = \sum_{e_i \in e(a)} P(e_i)$$

# Independence

Critical property! But rare.

If A and B are independent:

- P(A and B) = P(A)P(B)
- P(A or B) = P(A) + P(B) - P(A)P(B)

Can break joint prob. table into separate tables.

# Independence

Are *Raining* and *Cold* independent?

| **Raining** | **Cold** | **Prob.** |
|---|---|---|
| True | True | 0.3 |
| True | False | 0.1 |
| False | True | 0.4 |
| False | False | 0.2 |

P(Raining) = 0.4
P(Cold) = 0.7

# Independence: Examples

*Independence: two events don't effect each other.*

- Duke winning NCAA, Dem winning presidency.
- Two successive, fair, coin flips.
- It raining, and winning the lottery.
- Poker hand and date.

Often we have an intuition about independence, but *always verify*. **Dependence does not mean causation!**

# Mutual Exclusion

Two events are mutually exclusive when:

- P(A or B) = P(A) + P(B).
- P(A and B) = 0.

This is *different from* independence.

# Independence is Critical

To compute P(A and B) we need a joint probability.

- This grows very fast.
- Need to sum out the other variables.
- Might require lots of data.
- NOT a function of P(A) and P(B).

If A and B are independent, then you can use separate, smaller tables.

*Much of machine learning and statistics is concerned with identifying and leveraging independence and mutual exclusivity.*

# Conditional Probabilities

What if you have a joint probability, and you *acquire new data?*

*My iPhone tells me that its cold. What is the probability that it is raining?*

| **Raining** | **Cold** | **Prob.** |
|---|---|---|
| True | True | 0.3 |
| True | False | 0.1 |
| False | True | 0.4 |
| False | False | 0.2 |

Write this as:

- P(Raining | Cold)

# Conditional Probabilities

We can write:

$$P(a|b) = \frac{P(a \text{ and } b)}{P(b)}$$

This tells us the probability of *a* **given only knowledge b**.

This is a probability w.r.t a **state of knowledge.**

- P(Disease | Symptom)
- P(Raining | Cold)
- P(Duke win | injury)

# Conditional Probabilities

DUKE COMPUTER SCIENCE

| Raining | Cold | Prob. |
|---|---|---|
| True | True | 0.3 |
| True | False | 0.1 |
| False | True | 0.4 |
| False | False | 0.2 |

P(Raining | Cold)
= P(Raining and Cold)
/ P(Cold)

… P(Cold) = 0.7
… P(Raining and Cold) = 0.3

P(Raining | Cold) ~= 0.43.

Note!

**P(Raining | Cold) + P(not Raining | Cold) = 1!**

# Bayes's Rule

Special piece of conditioning magic.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

If we have conditional P(B | A) and we **receive new data** for B, we can compute **new distribution for A.** (Don't need joint.)

**As evidence comes in, revise belief.**

# Bayes Example

Suppose P(cold) = 0.7, P(headache) = 0.6.
P(headache | cold) = 0.57

What is P(cold | headache)?

$$P(c|h) = \frac{P(h|c)P(c)}{P(h)}$$

$$P(c|h) = \frac{0.57 \times 0.7}{0.6} = 0.66$$

Not always symmetric!
Not always intuitive!