Due on March 8th, 2016
120 points total

General Directions: If you are asked to provide an algorithm, you should clearly define each step of the procedure, establish its correctness, and then analyze its overall running time. There is no need to write pseudo-code; an unambiguous description of your algorithm in plain text will suffice.

All the answers must be typed, preferably using LaTeX. If you are unfamiliar with LaTeX, you are strongly encouraged to learn it. However, answers typed in other text processing software and properly converted to a pdf file will also be accepted. Before submitting the pdf file, please make sure that it can be opened using any standard pdf reader (such as Acrobat Reader) and your entire answer is readable. **Handwritten answers or pdf files that cannot be opened will not be graded and will not receive any credit.**

Finally, please read the detailed collaboration policy given on the course website. You are **not** allowed to discuss homework problems in groups of more than 3 students. **Failure to adhere to these guidelines will be promptly reported to the relevant authority without exception.**

Problem 1 (20 points)
You have been called to mediate in an "algorithmic dispute" between Alice Algorithmix and Bob Bitfiddler! Alice and Bob have been tasked by a courier company to find routes in Durham such that each destination is reached using the shortest route starting at their delivery center. The company wants to save overhead; therefore, they require that the roads traversed in all the routes should be acyclic (call this a shortest path tree). Having learnt minimum spanning tree (MST) algorithms from you, Alice has found an MST on the Durham roadmap (think of roads as edges and intersections/destinations as vertices) and claims that for every graph, there always exists an MST that is also a shortest path tree. On the other hand, Bob claims that it is possible to have an MST and a shortest path tree that are completely edge-disjoint. (Assume all roads are bi-directional, i.e., the graph is undirected. Also, assume that all edge lengths are positive and distinct.)

(a) (10 points) Is Alice's claim correct? Give a proof for your answer.

(b) (10 points) Is Bob's claim correct? Give a proof for your answer.

Problem 2 (5 points)
Show that the augmenting paths algorithm for finding a maximum flow will converge in finite time if all edge capacities are rational (recall that a rational number is one that can be written in the form a/b where a and b are integers).

Problem 3 (15 points)
Duke is hosting a conference for which the organizers want to ensure that traffic moves smoothly from the hotel where the delegates will be staying to the conference venue. After taking into account existing traffic patterns, the city has set limits on the number of cars carrying delegates that can pass through each intersection (the roads themselves have no limit). Can you help the conference organizers decide the maximum number of delegates they can invite? (Assume that each delegate will use a separate car. Roads can be one-way or two-way.)

Problem 4 (25 points)
Since you are taking this course in February, everyone in class is camping out for the UNC game. After lecture, all of you need to get to your tents in Krzyzewskiville as fast as possible. Being the all-star student that you are, everyone turns to you to design an algorithm that will find the fastest way to route everyone in the lecture to the campsite. In other words, you goal is to minimize the latest arrival time over all students.

You bring up a map of Duke on your smartphone consisting of walkways and intersections between the walkways. For simplicity, assume that any segment of a walkway connecting two intersections takes unit time to traverse, and students can only leave the lecture hall at integer times (so if the lecture ends at time $t = 0$, then students can leave at times $t = 0, 1, 2,$, etc.). For each of these walkway segments, the map also specifies its pedestrian capacity, i.e., if the walkway segment that connects intersections a and b has pedestrian capacity u, then you can send up to u people from a to b in a unit amount of time (and vice versa).

Design an algorithm to figure out the minimum time in which all of you can reach Krzyzewskiville.

Hint: Use maximum flows. Let T be the set of times and V be the set of intersections. Consider a network with vertices $T \times V$ (this denotes the Cartesian product of T and V; look up this notation if it is unfamiliar). What should the edges of this network be, and how can you use it to solve the problem?

Problem 5 (55 points)

Let $G = (V, E)$ be an directed graph where $u : E \to \mathbb{Z}_+$ and $\ell : E \to \mathbb{Z}_+$ are functions that give non-negative integer upper and lower capacities for each edge, where $\ell(x, y) \leq u(x, y)$ for all $(x, y) \in E$. In addition to the usual flow constraints (capacity constraints and flow balance constraints), a feasible flow on this graph must also satisfy the property that the flow on an edge is at least its lower capacity.

(a) (5 points) Give an example of a graph with no feasible s-t flow.

(b) (20 points) Give an algorithm for finding a feasible flow from s to t (if it exists). *Hint: First, send flow on each edge equal to its lower capacity. Note that this can violate balance constraints. Argue that the total flow deficit is equal to the total flow surplus, including the s, t imbalance. Finally, create a new network and show that finding a max flow of a certain value on this new network solves for a flow that is now feasible .*

(c) (15 points) Show how to find a minimum value feasible flow from the feasible flow you found in part (b). *Hint: Slightly alter the definition of residual networks to solve this problem.*

(d) (15 points) Define the *capacity lower-bound* of an s-t cut S to be:

$$\ell(S) = \sum_{x \in S, y \in \overline{S}} \ell(x, y) - \sum_{x \in \overline{S}, y \in S} u(x, y).$$

Show that the minimum value of a feasible flow from s to t is equal to the maximum capacity lower-bound of a cut separating s from t.