Algorithms in the Real World

Linear and Integer Programming I

- Introduction
- Geometric Interpretation
- Simplex Method
- Dual Formulation

Linear and Integer Programming

Linear or Integer programming (one formulation)

find x to minimize $z = c^T x$ **cost** or **objective** function

subject to $Ax \leq b$ **inequalities**

$x \geq 0$

$c \in R^n, \quad b \in R^m, \quad A \in R^{n \times m}$

Linear programming:

$x \in R^n$ (polynomial time)

Integer programming:

$x \in Z^n$ (NP-complete)

Extremely general framework, especially IP

Related Optimization Problems

Unconstrained optimization

$\min\{f(x) : x \in R^n\}$

(I denotes inequality, E denotes equality.)

Constrained optimization

$\min\{f(x) : c_i(x) \leq 0, i \in I, c_j(x) = 0, j \in E\}$

Quadratic programming

$\min\{(x^TQx)/2 + c^Tx : a_i^Tx \leq b_i, i \in I, a_i^Tx = b_j, j \in E\}$

Zero-One programming

$\min\{c^Tx : Ax = b, x \in \{0,1\}^n, c \in R^n, b \in R^m\}$

Mixed Integer Programming

$\min\{c^Tx : Ax \leq b, x \geq 0, x_i \in Z^n, i \in \text{Int\_Variables}, x_r \in R^n, r \in \text{Real\_Variables}\}$

How important is optimization?

- 50+ packages available
- 1300+ papers just on interior-point methods
- 100+ books in the library
- 10+ courses at most universities
- 100s of companies
- All major airlines, delivery companies, trucking companies, manufacturers, etc…
  make serious use of optimization.

Linear+Integer Programming Outline

Linear Programming

- General formulation and geometric interpretation
- Simplex method
- Ellipsoid method
- Interior point methods

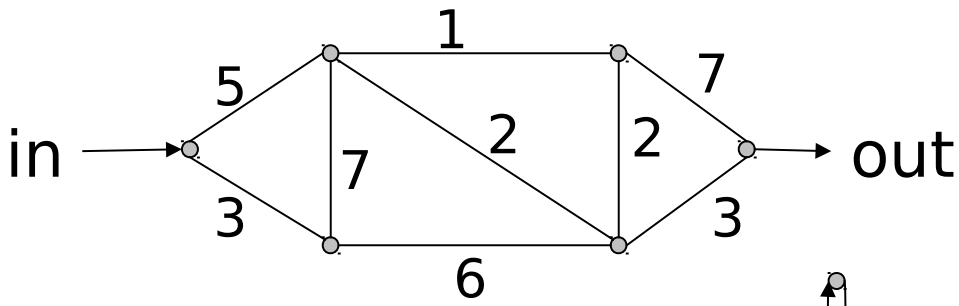
Integer Programming

- Various reductions from NP hard problems
- Linear programming approximations
- Branch-and-bound + cutting-plane techniques
- Case study from Delta Airlines

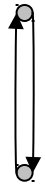
Applications of Linear Programming

1. A substep in most integer and mixed-integer linear programming (MIP) methods
2. Selecting a mix: oil mixtures, portfolio selection
3. Distribution: how much of a commodity should be distributed to different locations.
4. Allocation: how much of a resource should be allocated to different tasks
5. Network Flows

Linear Programming for Max-Flow

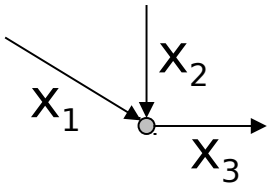


Create <u>two variables per edge</u>: x_1 x_1'



Create <u>one equality per vertex</u>:

$$x_1 + x_2 + x_3' = x_1' + x_2' + x_3$$



and <u>two inequalities per edge</u>:

$$x_1 \le 3, \quad x_1' \le 3$$

add <u>edge x_0 from **out** to **in**</u>

<u>**maximize**</u> x_0

In Practice

In the “real world” most problems involve at least some integral constraints.

- Many resources are integral
- Can be used to model yes/no decisions (0-1 variables)

Therefore “1. A substep in integer or MIP programming” is the most common use in practice

Algorithms for Linear Programming

- **Simplex** (Dantzig 1947)
- **Ellipsoid** (Kachian 1979)
  first algorithm known to be **polynomial time**
- **Interior Point**
  first practical polynomial-time algorithms
  - **Projective method** (Karmakar 1984)
  - **Affine Method** (Dikin 1967)
  - **Log-Barrier Methods** (Frisch 1977, Fiacco 1968, Gill et al. 1986)

Many of the interior point methods can be applied to nonlinear programs. (But not necessarily in poly time.)

State of the art

1 million variables

10 million nonzeros

No clear winner between Simplex and Interior Point

- Depends on the problem
- Interior point methods are subsuming more and more cases
- All major packages supply both

The truth: the sparse matrix routines, make or break both methods.

The best packages are highly sophisticated.

Formulations

There are many ways to formulate linear programs:

- **objective (or cost) function**
  minimize $c^T x$, or
  maximize $c^T x$, or
  find any feasible solution
- **(in)equalities**
  $Ax \leq b$, or
  $Ax \geq b$, or
  $Ax = b$, or any combination
- **nonnegative variables**
  $x \geq 0$, or not

Fortunately it is pretty easy to convert among forms

Formulations

The two **most common** formulations:

Canonical form		Standard form
minimize $c^T x$	**slack** variables →	minimize $c^T x$
subject to $Ax \geq b$		subject to $Ax = b$
$x \geq 0$		$x \geq 0$

e.g.

$7x_1 + 5x_2 \geq 7$	y_1 →	$7x_1 + 5x_2 - y_1 = 7$
$x_1, x_2 \geq 0$		$x_1, x_2, y_1 \geq 0$

More on slack variables later.

Geometric View of Canonical Form

A **polytope** in n-dimensional space

- Each inequality corresponds to a half-space.
- The “feasible set” is the intersection of the half-spaces
- This corresponds to a polytope
- Polytopes are **convex**: if x,y is in the polytope, so is the line segment joining them.
- The optimal solution is at a vertex (i.e., a corner).

Geometric View of Canonical Form

minimize:

$$z = -2x_1 - 3x_2$$

subject to:

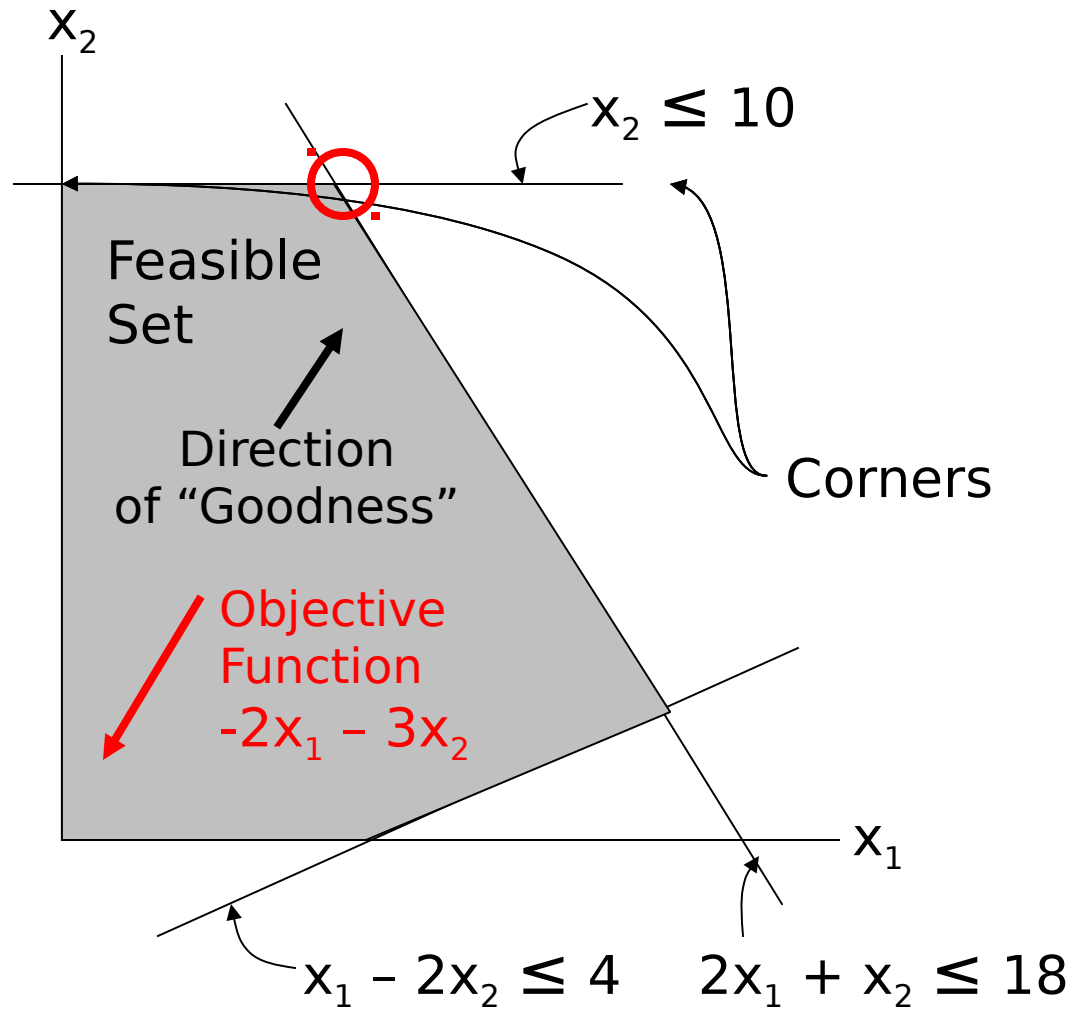
$$x_1 - 2x_2 \le 4$$

$$2x_1 + x_2 \le 18$$

$$x_2 \le 10$$

$$x_1, x_2 \ge 0$$

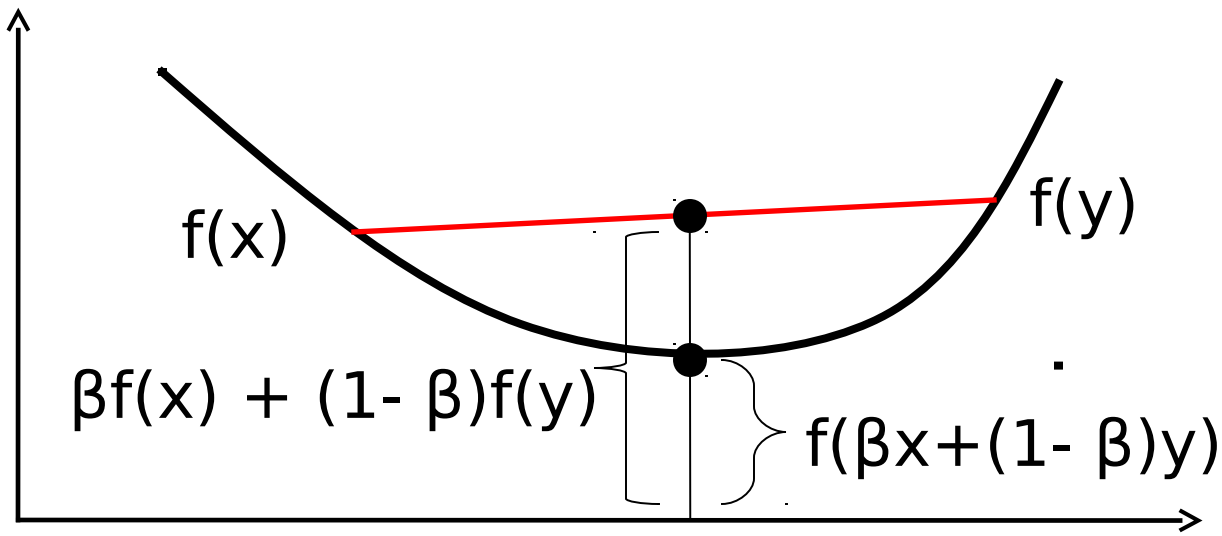
An intersection of 5 halfspaces



Convex Function

f is convex if for all vectors x, y $\in$ S and $\beta \in [0,1]$

$f(\beta x+(1-\beta)y) \leq \beta f(x) + (1-\beta)f(y)$

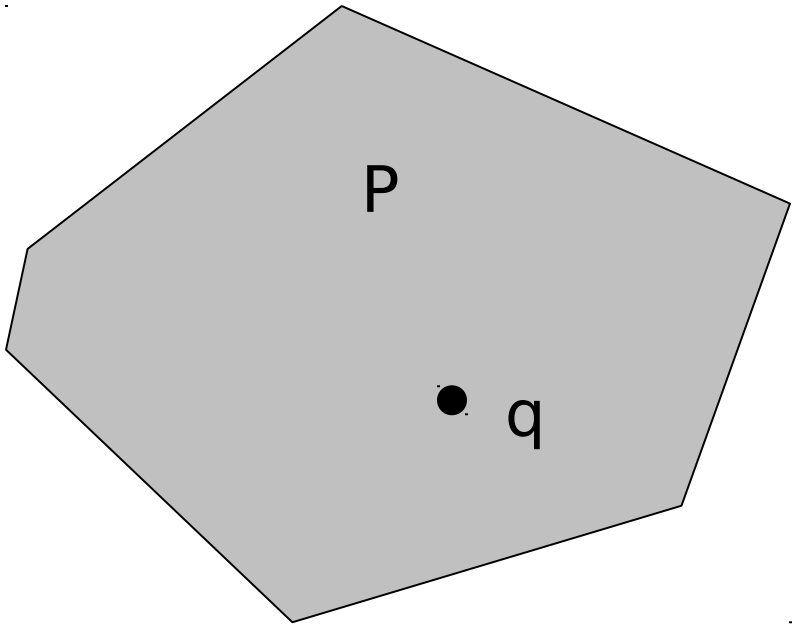


(Note that linear cost functions are both convex and concave.)

"Fundamental Theorem of Linear Programming"

The optimum is at a vertex.

Vertices v_i



If the polytope P is closed, every point q in P is a convex combination of vertices of P, i.e., there exist β_i such that $q = \sum \beta_i v_i$

If f is convex, then

$$f(q) = f(\sum \beta_i v_i) \le \sum \beta_i f(v_i) \le \max_i f(v_i)$$

(for max problem, optimum is at a vertex)

If f is linear, then

$$f(q) = f(\sum \beta_i v_i) = \sum \beta_i f(v_i)$$

so

$$\min_i f(v_i) \le f(q) \le \max_i f(v_i)$$

Geometric View of Canonical Form

A **polytope** in n-dimensional space

- Each inequality corresponds to a half-space.
- The “feasible set” is the intersection of the half-spaces.
- This corresponds to a polytope
- The optimal solution is at a corner.

Simplex moves around on the surface of the polytope

Interior-Point methods move within the polytope

Notes about higher dimensions

In canonical form $Ax \leq b$ or standard form $Ax = b$, (m equations and n variables)

Each corner (extreme point) consists of:

- At least n intersecting (n-1)-dimensional **hyperplanes**

  (hyperplane: $H=\{x: a_1x_1 + \ldots + a_nx_n = b\}$)

  i.e., n linearly independent constraints are tight at the corner

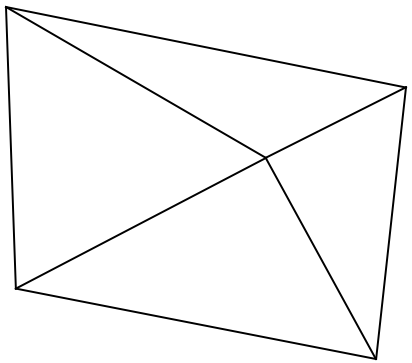
  ("degenerate" if more than n hyperplanes intersect)

- e.g., in 2 dimensions, hyperplanes are lines, and a corner is the intersection of 2 or more lines
- in 3 dimensions, hyperplanes are planes, and a corner is the intersection of 3 or more planes
- At least n **edges** intersect at a corner
- Each edge corresponds to moving off of one hyperplane (still constrained by at least n-1 of them)

Simplex moves from corner to corner along the edges

The Simple Essense of Simplex

Polytope P



Input: min $f(x) = cx$
s.t. x in $P = \{x: Ax \leq b, x \geq 0\}$

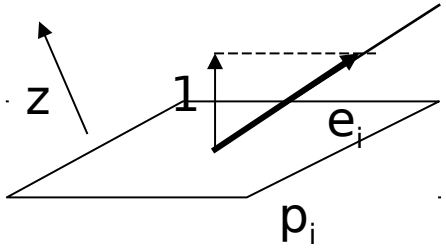
Consider Polytope P from canonical form as a graph $G = (V,E)$ with
V = polytope vertices,

1) Find *any* vertex v of P.
2) While there exists a neighbor u of v in G with $f(u) < f(v)$, update v to u.
3) Output v.

Choice of neighbor if several u have $f(u) < f(v)$?
Termination? Correctness? Running Time?

Optimality and Reduced Cost

The **reduced cost** for a hyperplane at a corner is the cost of moving one unit away from the plane along its corresponding edge.



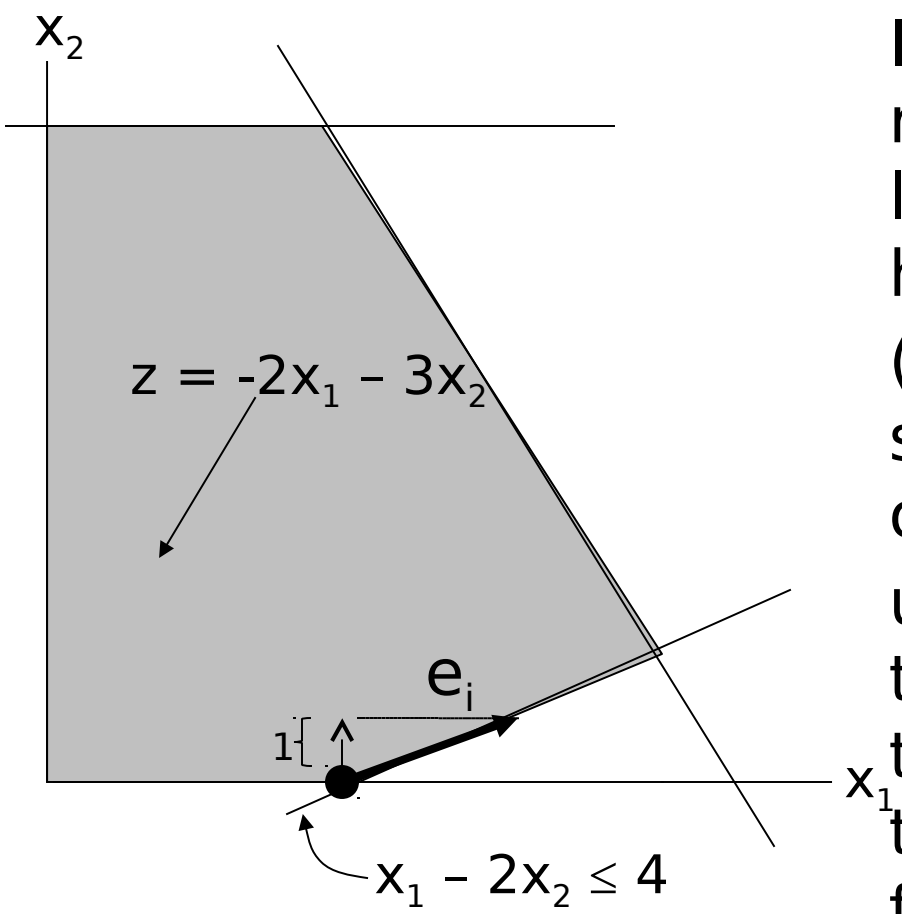
$$r_i = c \bullet e_i$$

(e_i is the distance traveled on edge for plane p_i, c is the cost function)

For **minimization**, if all reduced costs are non-negative, then we are at an optimal solution.

Finding the most negative reduced cost is one often-used heuristic for choosing an edge to leave on

Reduced cost example



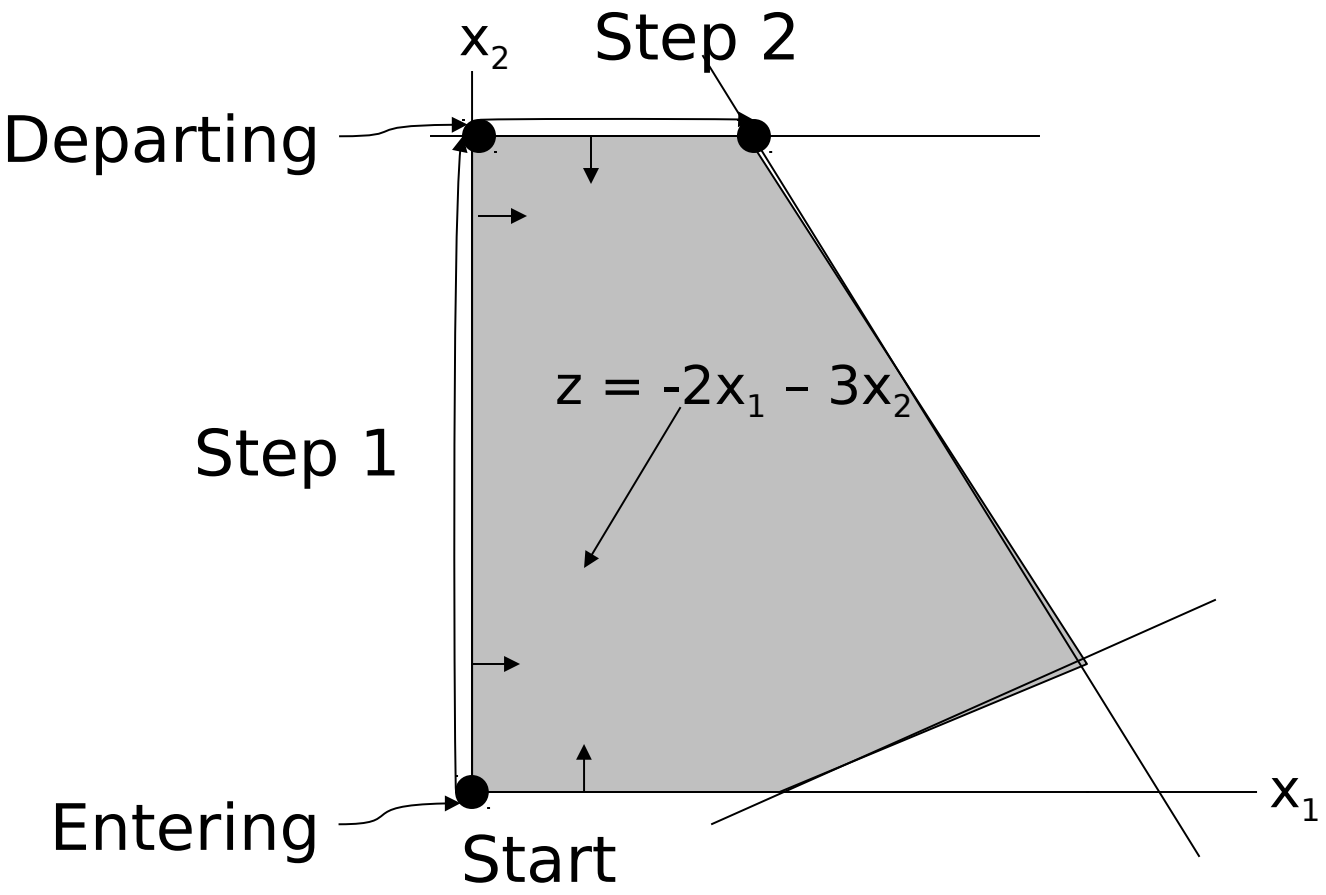
$e_i = (2, 1)$

In the example the reduced cost of leaving the hyperplane $x_2=0$ is $(-2,-3) \bullet (2,1) = -7$ since moving one unit off of $x_2=0$ will move us (2,1) units along the edge. We take the dot product of this and the cost function.

Simplex Algorithm

1. Find a **corner of the feasible region**
2. **Repeat**
   A. For each of the n hyperplanes intersecting at the corner, calculate its **reduced cost**
   B. If they are all non-negative, then **done** (proof later)
   C. Else, pick the most negative reduced cost
      This is called the **entering** plane
   D. Move along corresponding edge (i.e., leave that hyperplane) until we reach the next corner (i.e., reach another hyperplane)
      The new plane is called the **departing** plane

Example



Simplifying

Problem:

- The $Ax \leq b$ constraints not symmetric with the $x \geq 0$ constraints.
  We would like more symmetry.

Idea:

- Make all inequalities of the form $x \geq 0$.

Use "slack variables" to do this.

Convert into form:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Standard Form

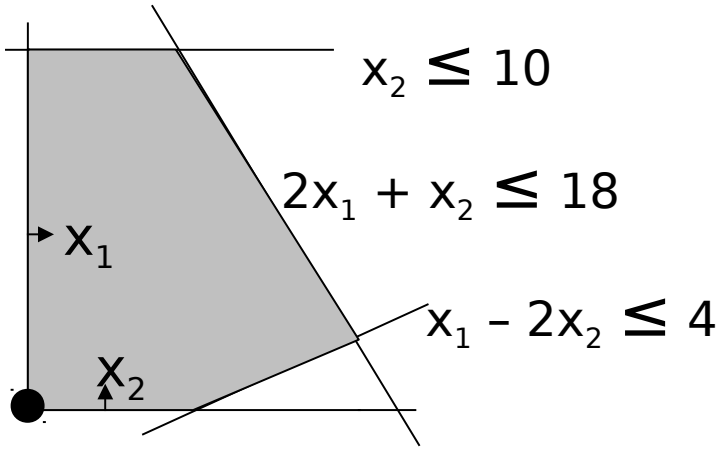
Standard Form

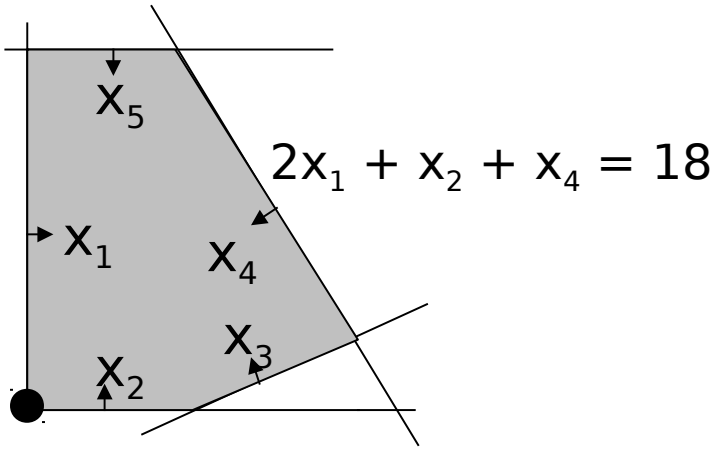
minimize $c^T x$	minimize $c^T x'$
subject to $Ax \le b$	subject to $A'x' = b$
$x \ge 0$	$x' \ge 0$

slack variables →

$|A| = m \times n$
i.e., m inequalities, n variables

$|A'| = m \times (m+n)$
i.e., m equations, m+n variable





Example, again

minimize:

$z = -2x_1 - 3x_2$

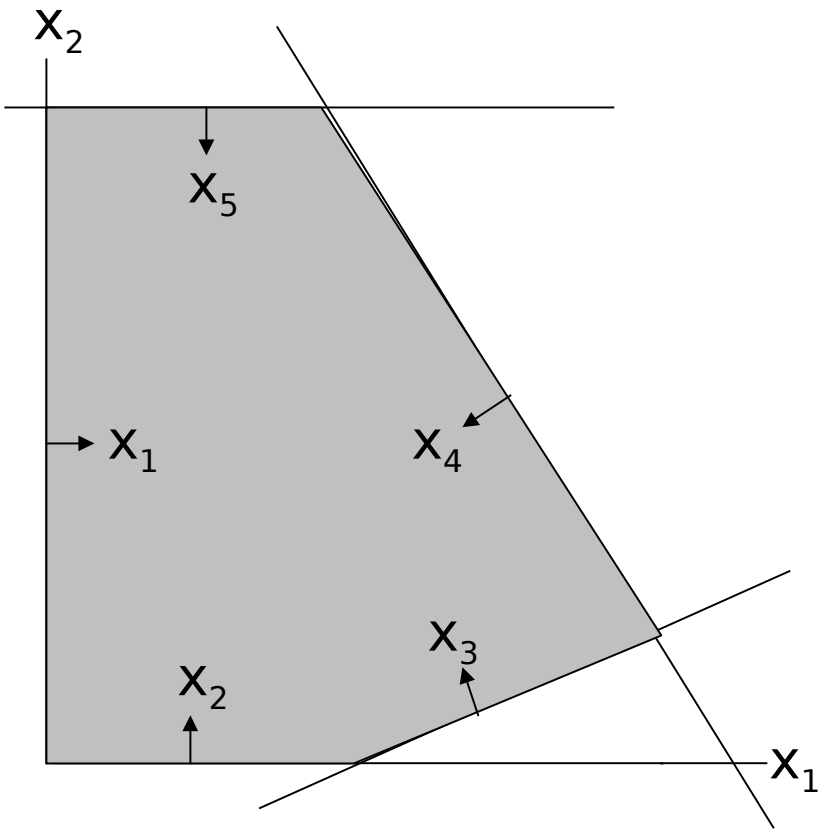
subject to:

$x_1 - 2x_2 + \underline{x_3} = 4$

$2x_1 + x_2 + \underline{x_4} = 18$

$x_2 + \underline{x_5} = 10$

$x_1, x_2, \underline{x_3}, \underline{x_4}, \underline{x_5} \geq 0$

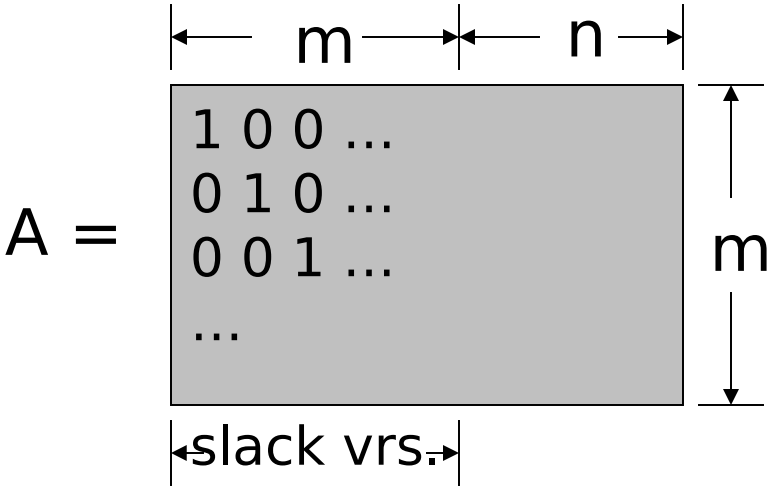


The equality constraints impose a 2d plane (a 2d subspace) embedded in 5d space, looking at the plane gives the figure above

Using Matrices

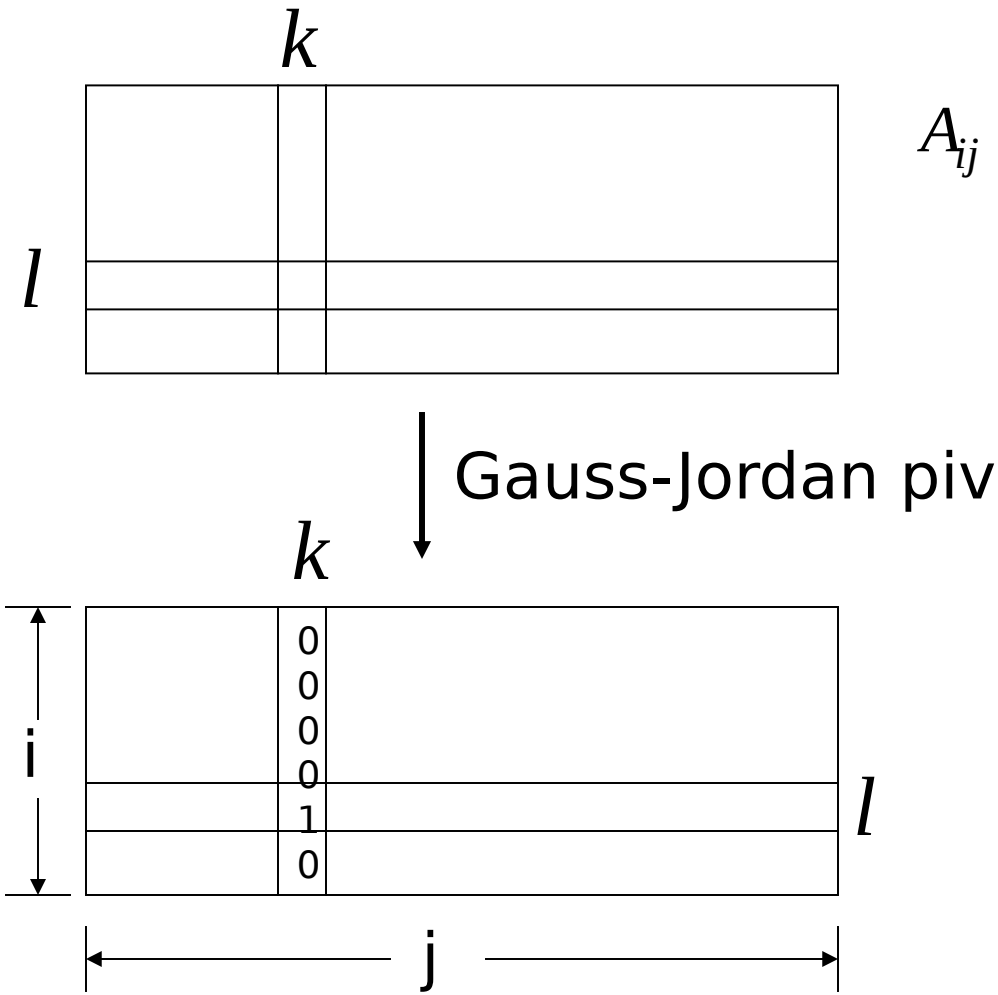
If before adding the slack variables A has size m x n (m inequalities, n variables), then after it has size m x (n + m)

m can be larger or smaller than n



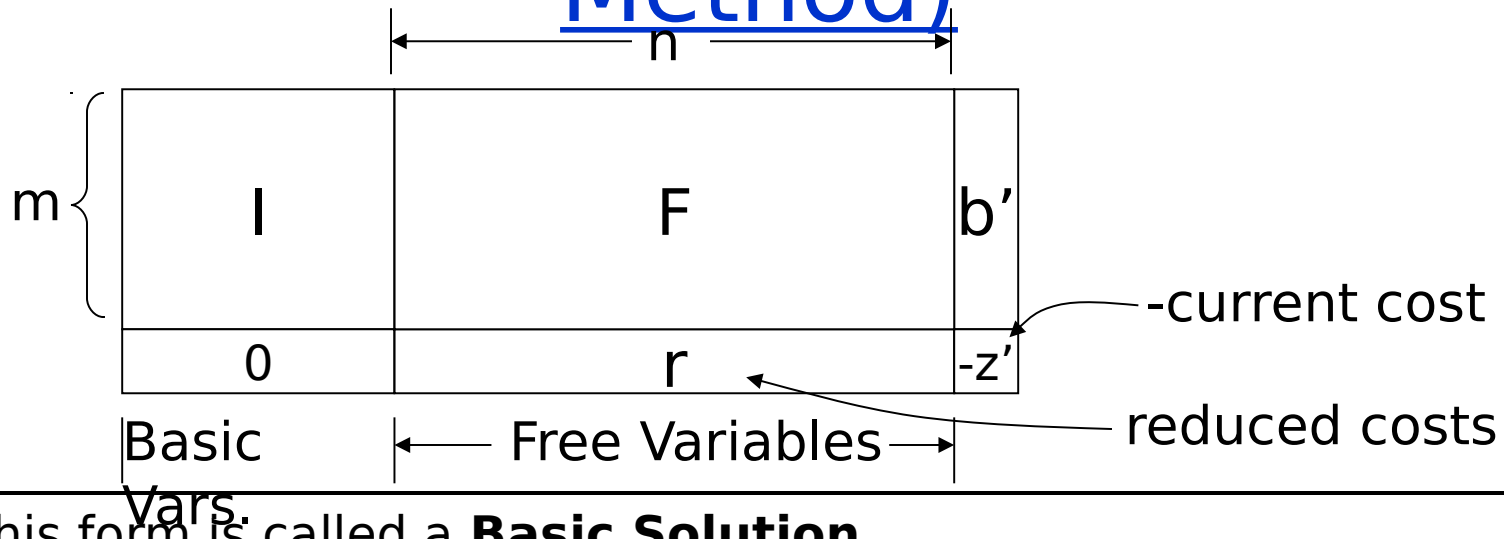
Assuming rows are independent, the solution space of Ax = b is an n-dimensional subspace on n+m variables.

Gauss-Jordan Elimination



$$B_{ij} = \begin{cases} A_{ij} - A_{lj}\dfrac{A_{ik}}{A_{lk}} & i \neq l \\ \dfrac{A_{lj}}{A_{lk}} & i = l \end{cases}$$

Simplex Algorithm (Tableau Method)



This form is called a **Basic Solution**

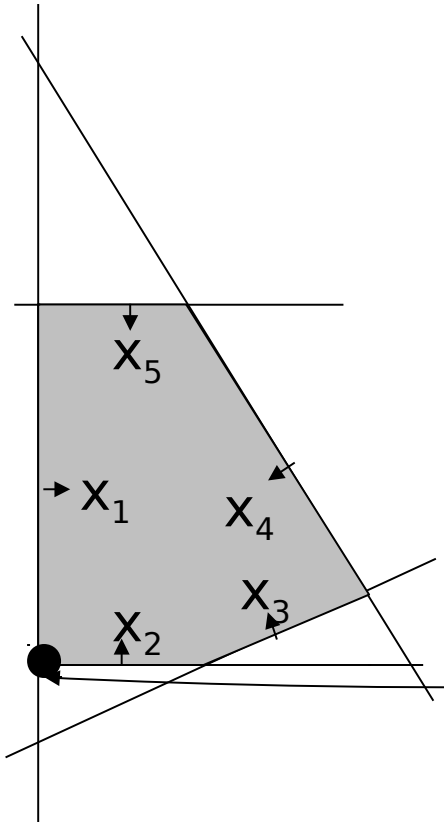
- the n "**free**" variables are set to 0
- the m "**basic**" variables are set to b'

A valid solution to Ax = b if I, F, b' reached using pivoting and column swapping from the original system A, b

Represents n intersecting hyperplanes

If feasible (i.e., b' $\geq$ 0), then the solution is called a **Basic Feasible Solution** and is a corner of the feasible set

Corner

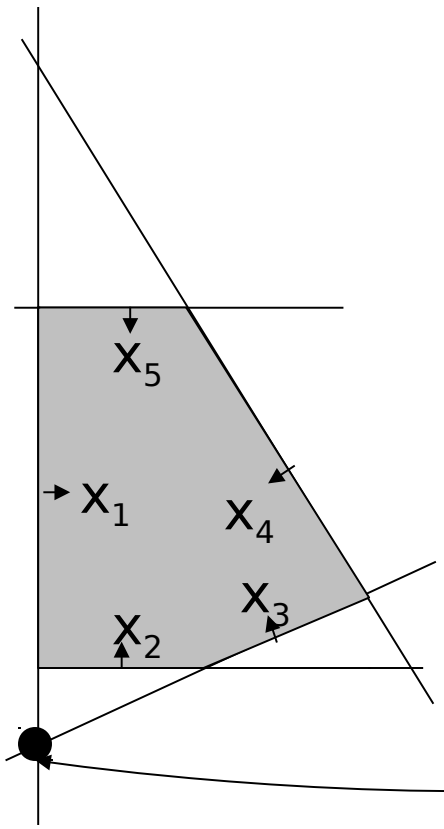


basic variables free variables

1	0	0	1	-2	4
0	1	0	2	1	18
0	0	1	0	1	10
0	0	0	-2	-3	0
x_3	x_4	x_5	x_1	x_2	

free variables indicate corner

Corner

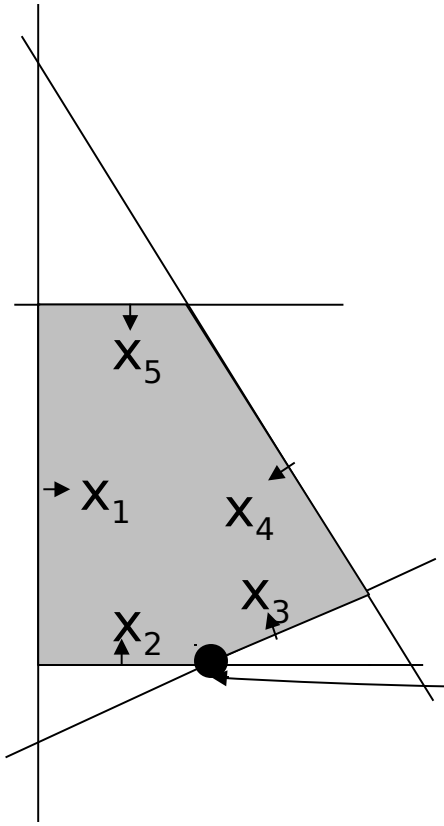


1	0	0	-.5	-1	-2
0	1	0	2.5	1	20
0	0	1	.5	1	12
0	0	0	-3.5	-3	-6
x_2	x_4	x_5	x_1	x_3	

Exchange free/basic variables by swapping columns, using GE to restore to tableau format.

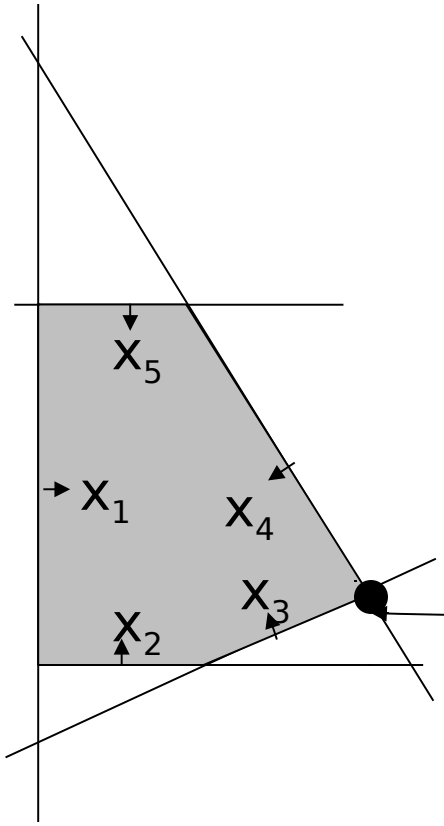
(“corner” not necessarily feasible, but we won’t allow this move)

Corner



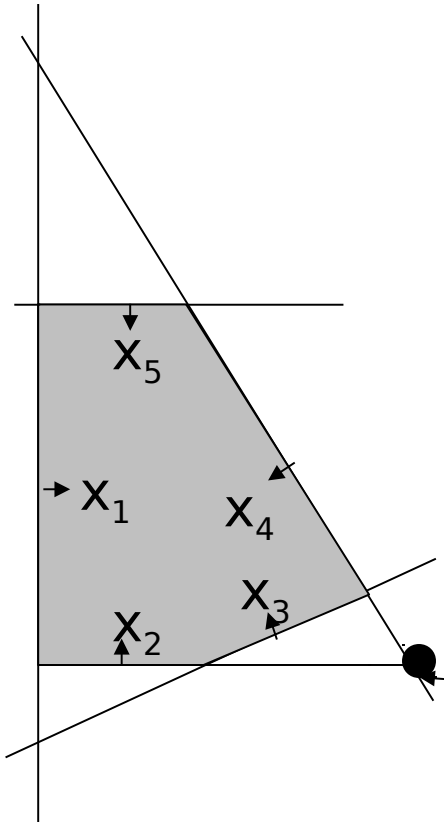
1	0	0	1	-2	4
0	1	0	-2	5	10
0	0	1	0	1	10
0	0	0	2	-7	8
x_1	x_4	x_5	x_3	x_2	

Corner



1	0	0	.2	.4	8
0	1	0	-.4	.2	2
0	0	1	.4	-.2	8
0	0	0	-.8	1.4	22
x_1	x_2	x_5	x_3	x_4	

Corner



1	0	0	.5	-2.5	-5
0	1	0	.5	.5	9
0	0	1	0	1	10
0	0	0	1	-2	18
x_3	x_1	x_5	x_4	x_2	

Note that in general there are n+m choose m corners

Simplex Method Again

Once you have found a basic feasible solution (a corner), we can move from corner to corner by swapping columns and eliminating.

ALGORITHM

1. Find a **basic feasible solution**
2. **Repeat**
   A. If r (reduced cost) ≥ 0 , DONE
   B. Else, pick column i with most negative r (nonbasic gradient heuristic)
   C. Pick row j with least non-negative b_j'/(j'th entry in column i)
   D. Swap columns
   E. Use Gaussian elimination to restore form

Tableau Method

A. If r are all non-negative then **done**

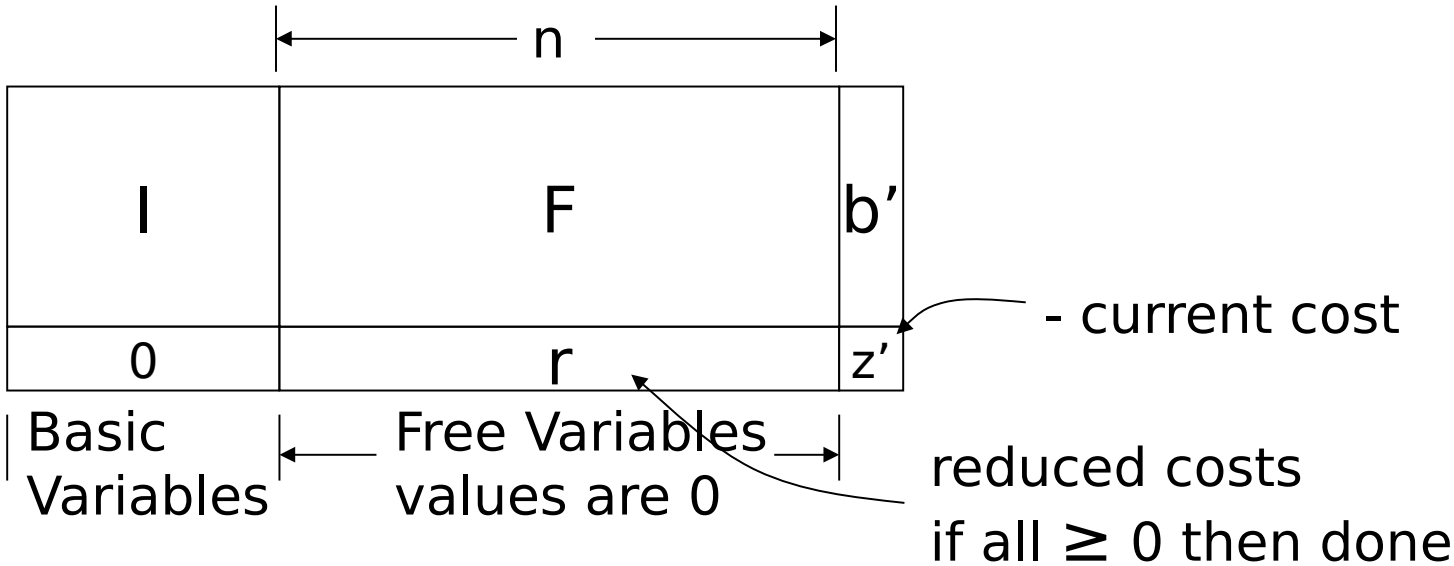


Tableau Method

B. Else, pick the most negative reduced cost
   This is called the **entering** plane

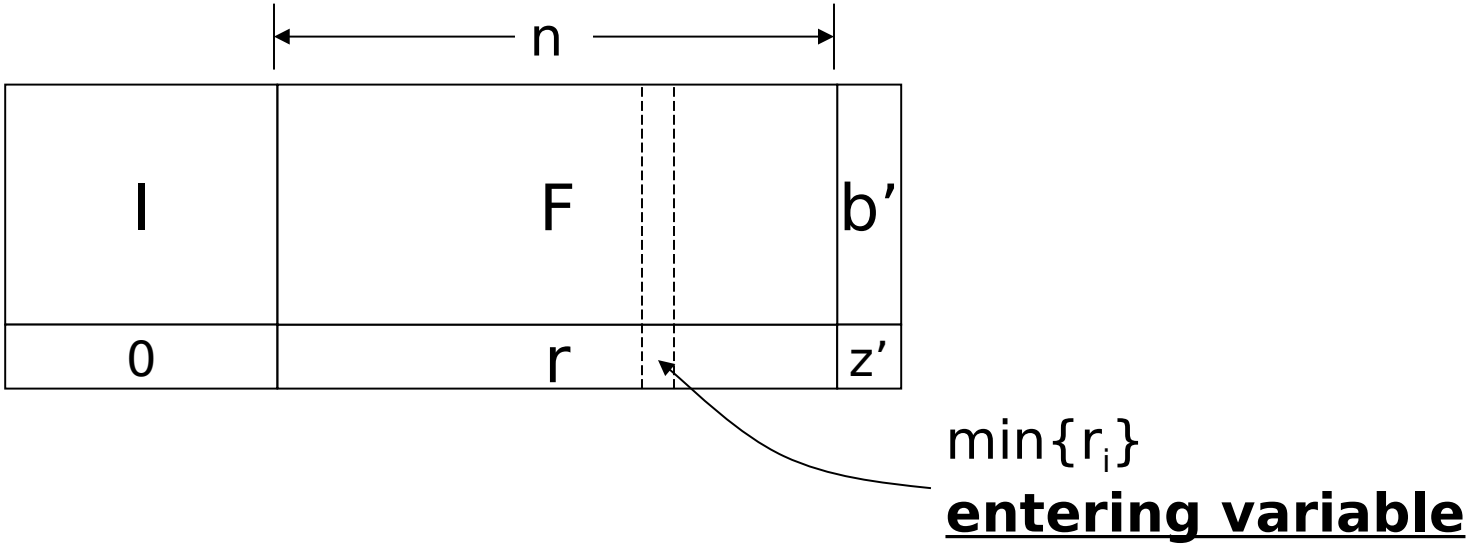


Tableau Method

C. Move along corresponding line (i.e., leave that hyperplane) until we reach the next corner (i.e. reach another hyperplane)
The new plane is called the **departing** plane

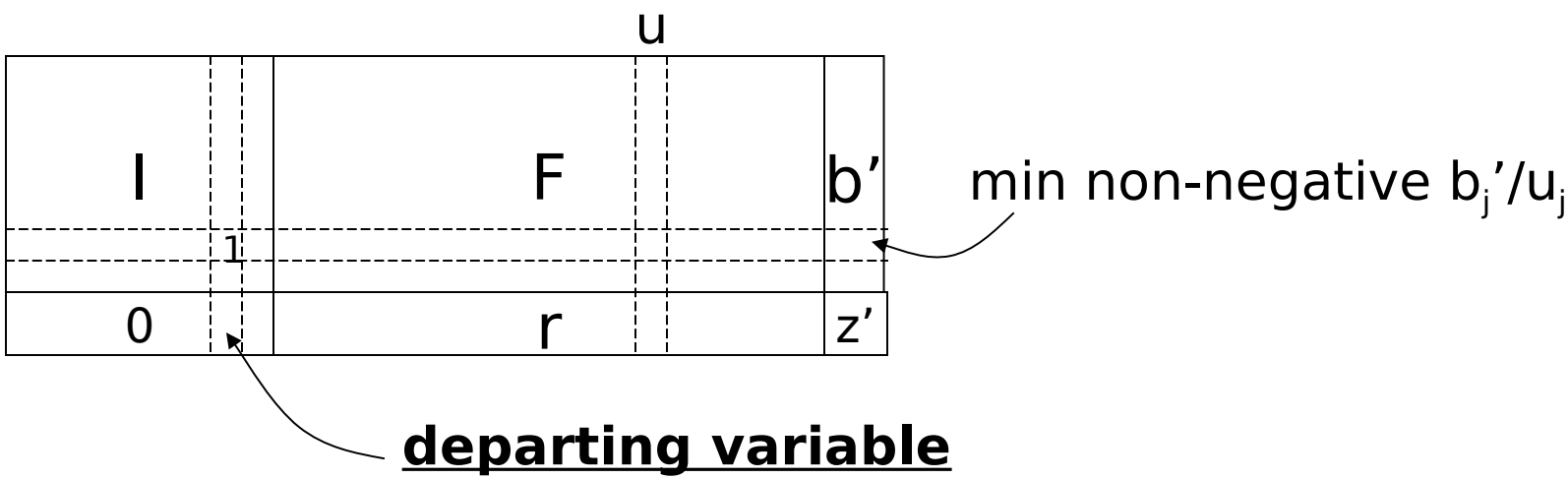
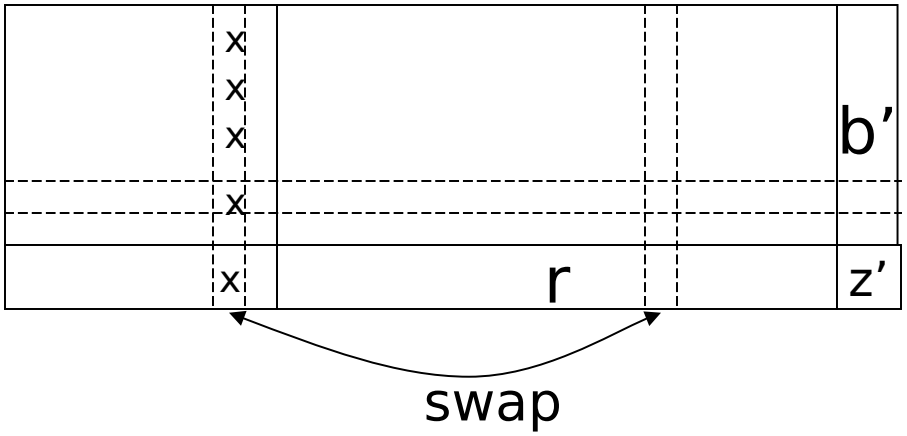


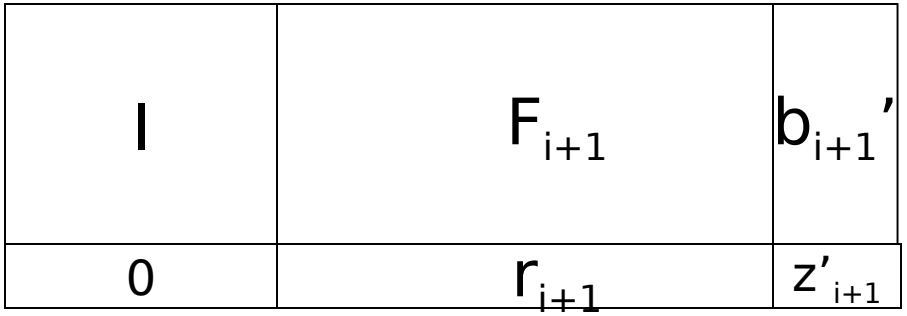
Tableau Method

D. Swap columns



No longer in proper form

E. Gauss-Jordan elimination



Back to proper form

Example

x_1	x_2	x_3	x_4	x_5	
1	-2	1	0	0	4
2	1	0	1	0	18
0	1	0	0	1	10
-2	-3	0	0	0	0
1	0	0	1	-2	4
0	1	0	2	1	18
0	0	1	0	1	10
0	0	0	-2	-3	0
x_3	x_4	x_5	x_1	x_2	

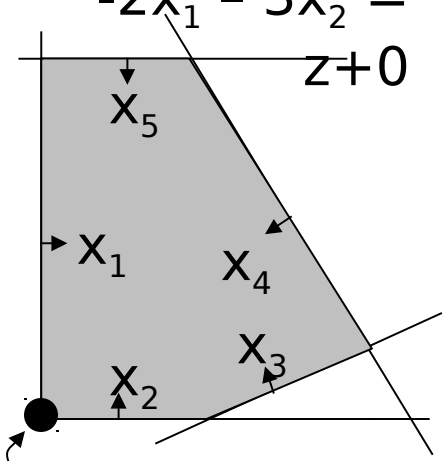
Find corner

$$x_1 - 2x_2 + x_3 = 4$$

$$2x_1 + x_2 + x_4 = 18$$

$$x_2 + x_5 = 10$$

$$-2x_1 - 3x_2 = z+0$$

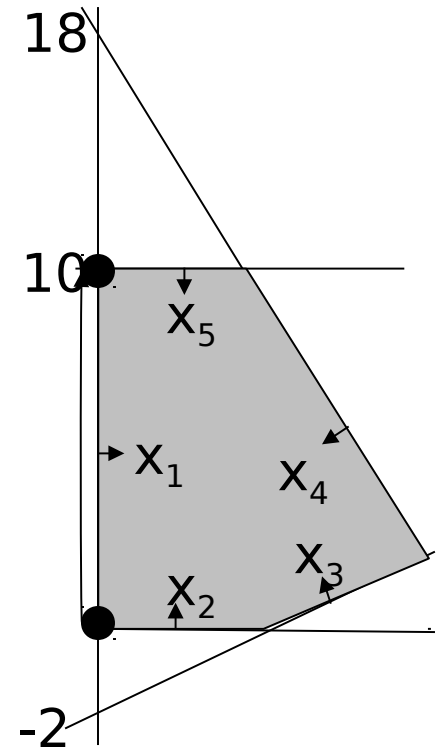


$x_1 = x_2 = 0$ (start)

Example

1	0	0	1	-2	4
0	1	0	2	1	18
0	0	1	0	1	10
0	0	0	-2	-3	0
x_3	x_4	x_5	x_1	x_2	

1	0	0	1	-2	4	-2
0	1	0	2	1	18	18
0	0	1	0	1	10	10
0	0	0	-2	-3	0	
x_3	x_4	x_5	x_1	x_2		b_j/v_j

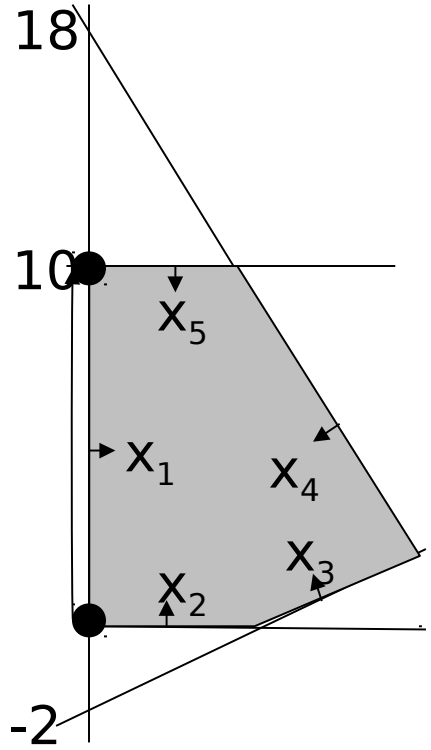


min
non-negative

Example

swap

1	0	-2	1	0	4
0	1	1	2	0	18
0	0	1	0	1	10
0	0	-3	-2	0	0
x_3	x_4	x_2	x_1	x_5	



Gauss-Jordan Elimination

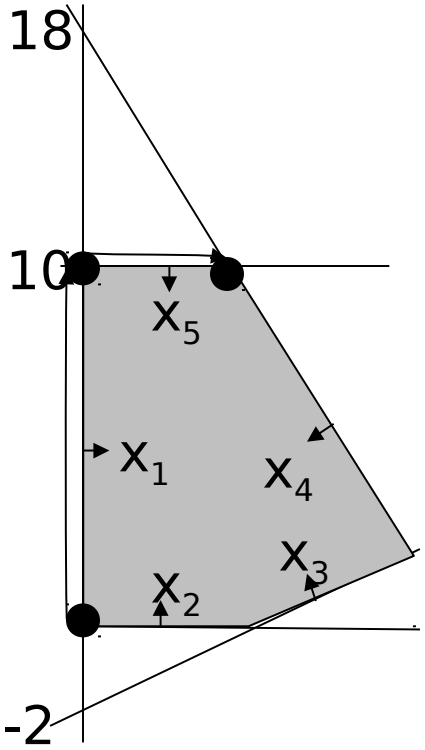
1	0	0	1	2	24
0	1	0	2	-1	8
0	0	1	0	1	10
0	0	0	-2	3	30
x_3	x_4	x_2	x_1	x_5	

$$-2x_1 + 3x_5 = z + 30 = z + z'$$

Example

1	0	0	1	2	24
0	1	0	2	-1	8
0	0	1	0	1	10
0	0	0	-2	3	30
x_3	x_4	x_2	x_1	x_5	

1	0	0	1	2	24	24
0	1	0	2	-1	8	4
0	0	1	0	1	10	-
0	0	0	-2	3	30	
x_3	x_4	x_2	x_1	x_5		

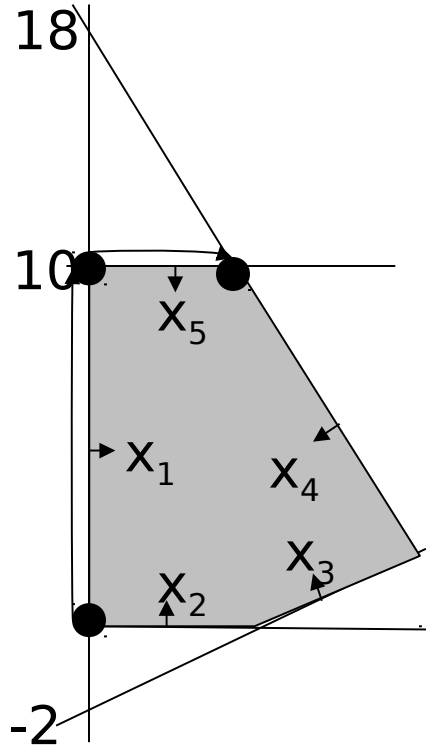


Example

swap

1	1	0	0	2	24
0	2	0	1	-1	8
0	0	1	0	1	10
0	-2	0	0	3	30
x_3	x_1	x_2	x_4	x_5	

1	0	0	-.5	2.5	20
0	1	0	.5	-.5	4
0	0	1	0	1	10
0	0	0	1	2	38
x_3	x_1	x_2	x_4	x_5	



Gauss-Jordan Elimination

Linear Algebra Proof that Local Min is Optimum

- Last row of tableau specifies a derived equation for cost, z, that holds for all feasible solutions.
- On termination, all coefficients in the equation are non-negative.
- E.g., from previous slide:

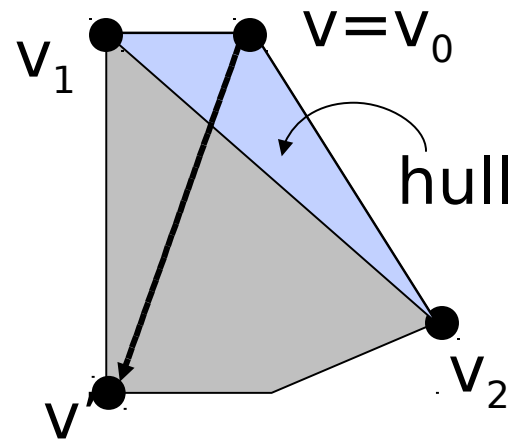
  $0x_3 + 0x_1 + 0x_2 + 1x_4 + 2x_5 = z + 38$, or

  $z = -38 + x_4 + 2x_5$
- Since x_4 and x_5 must be non-negative, minimum value of z must be -38.
- Thus, Simplex finds a global minimum by deriving a constraint on cost with all non-negative coefficients.

Geometric Proof that Local Min is Optimum

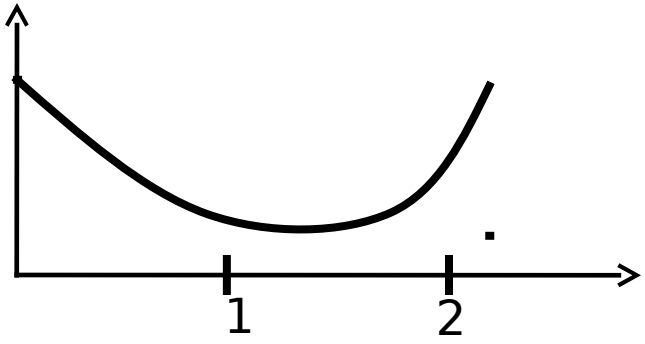
- Let $v=v_0$ be the corner at which Simplex stops, let $v_1, \ldots, v_n$ be the neighbors of v, and let f denote the cost function that is to be minimized. Thus $f(v) \leq f(v_i)$ for $0 \leq i \leq n$.
- Within the convex hull of $\{v_0,v_1,\ldots,v_n\}$, f takes on a minimum value at v, because for any q in the hull, $f(q) = f(\sum \beta_i v_i) = \sum \beta_i f(v_i)$ so $f(v) = \min_i f(v_i) \leq f(q)$

- Suppose there is a better solution at some corner v' outside of the hull. On the line segment from v to v', the cost function should decrease linearly, but this line segment must pass through the interior of the hull, and until it exits,



Minimizing / Maximizing Convex Cost Functions

- If trying to maximize a convex cost function over a closed convex space, the global maximum must occur at a corner, but a local maximum may not be a global maximum, e.g., maximize f(x) such that $1 \leq x \leq 2$



- If trying to minimize a convex cost function, a local minimum is always a global minimum, but it may not occur at a corner. (E.g., above.)
- Convex hull of neighbors proof shows Simplex stops at a local minimum. Since a linear cost function is convex, a local minimum is also a global minimum.

Problem?: Unbounded Solution

- What if all b_j/u_j are negative (i.e., all b_j are positive and all u_j negative)?
- Then there is no bounded solution.
- Can move an arbitrary distance from the current corner, reducing cost by an arbitrary amount.
- Not really a problem. LP is solved.

Problem: Cycling

- If the basic variable swapped in is already zero ($b'_j=0$), then the distance moved along the edge is 0.
- Although a basic variable is exchanged with a free variable, there is no reduction in cost and the algorithm stays at the same vertex.
- In this “degenerate” case more than n hyperplanes intersect at the vertex (more than n variables have value 0).
- The algorithm may cycle through classification of variables as basic and free and reach a tableau that has already been seen before! Infinite loop!
- Solution: Bland’s anticycling rule for tie breaking among columns & rows.

Simplex Concluding remarks

For dense matrices, takes O(n(n+m)) time per iteration

Can take an **exponential** number of iterations.

In practice, sparse methods are used for the iterations.

Spielman-Teng "Smoothed" Complexity

Theorem: If Gaussian noise is added to each coefficient of A and b, then the expected number of iterations required by the Simplex algorithm to solve $Ax \leq b$ (n variables and m inequalities) is polynomial in m and n and $1/\sigma$, where σ is the standard deviation of the random variable "maximum, taken over all individual coefficients, amount of noise added".

This theorem may provide a partial explanation for why the exponential worst-case running time of the Simplex algorithm is rarely seen in practice.

Duality

Primal (P):

maximize $z = c^T x$

subject to $Ax \leq b$

$x \geq 0$ (m equations, n variables)

Dual (D):

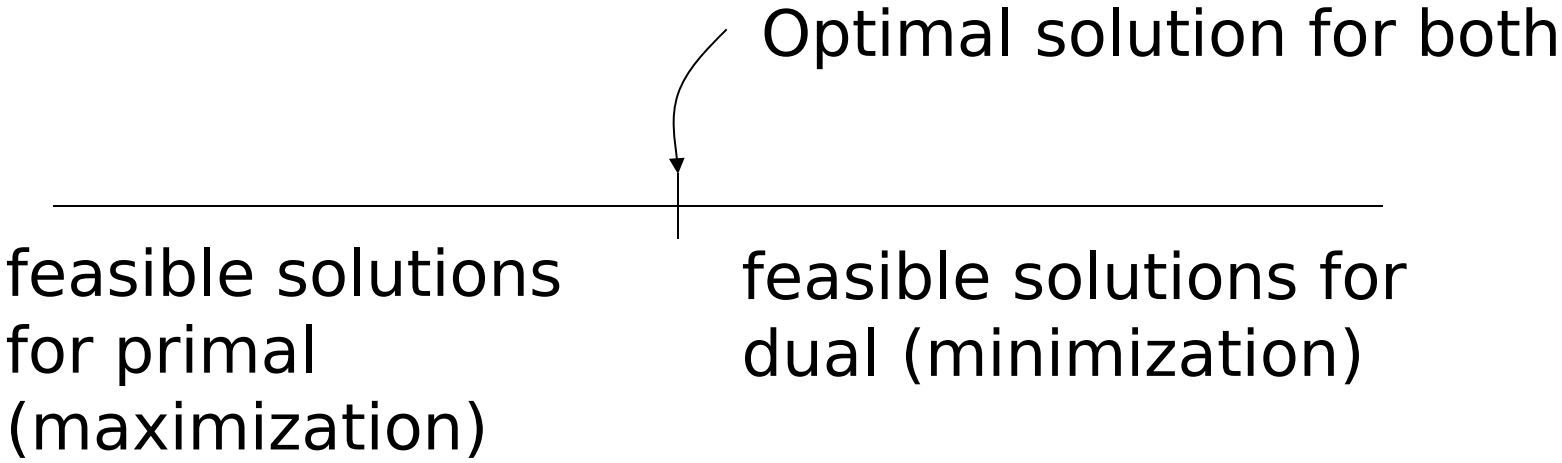
minimize $z = y^T b$

subject to $A^T y \geq c$

$y \geq 0$ (n equations, m variables)

Duality Theorem: if x is feasible for **P** and y is feasible for **D,** then $c^T x \leq y^T b$ and at optimality $c^T x = y^T b$.

Duality (cont.)



Quite similar to duality of Maximum Flow and Minimum Cut.

Useful in many situations.

Duality Example

Primal:

maximize:

$z = 2x_1 + 3x_2$

subject to:

$x_1 - 2x_2 \leq 4$

$2x_1 + x_2 \leq 18$

$x_2 \leq 10$

$x_1, x_2 \geq 0$

Dual:

minimize:

$z = 4y_1 + 18y_2 + 10y_3$

subject to:

$y_1 + 2y_2 \geq 2$

$-2y_1 + Y_2 + Y_3 \geq 3$

$y_1, y_2, y_3 \geq 0$

Solution to both is 38 (x_1=4, x_2=10), (y_1=0, y_2=1, y_3=2).