COMPSCI630 Randomized Algorithms
Assignment 1

Due Date: Mar. 11, 2016 in class.

Problem 1 (Problem 1.1 in Textbook)**.** Suppose you are given a coin for which the probability of HEADS, say p, is *unknown*. How can you use the coin to generate unbiased (i.e. $\Pr[HEADS] = \Pr[TAILS] = 1/2$) coin flips? Give a scheme for which the expected number of flips of the biased coin for extracting one unbiased coin-flip is no more than $1/[p(1-p)]$. (Hint: Consider two consecutive flips of the biased coin.)

Problem 2 (Card Collector)**.** Suppose you are playing a collectible card game. There are n cards in the game to be collected and your goal is to collect all of them. You can buy a uniformly random card for one dollar, or you can exchange four arbitrary cards for a specific card. Suppose you first buy random cards without exchanging, let T_m be the amount of money you have spent when you have collected m distinct cards. In this problem to simplify calculation we assume for any integers $0 < a < b$,

$$\sum_{i=a}^{b} 1/i = \log(b/a).$$

You can also look at `https://en.wikipedia.org/wiki/Geometric_distribution` for mean/variance of geometric distribution.

(a) Show $\mathbb{E}[T_m] = n\log(\frac{n}{n-m+1})$.

(b) Suppose $m < \lfloor \alpha n \rfloor$ for $\alpha < 1$, Show that $\mathrm{Var}[T_m] \le m/(1-\alpha)^2$.

(c) Now you can use trading. Show that there exists a constant c such that with probability at least 3/4, you can finish your collection after spending $cn \pm O(\sqrt{n})$ dollars. Try to estimate c.

Problem 3 (Problem 4.8 in Textbook, rephrased)**.** A *decision* problem is a problem whose output is either 0 or 1 (e.g. "Is this 3-SAT instance satisfiable?", "Is the MIN-CUT of this graph less than 5?"). For an input $x \in \{0,1\}^n$, suppose $P(x) \in \{0,1\}$ is the correct answer for the decision problem, and there is a randomized algorithm A that runs in polynomial time $T(n)$, and for any input x

$$\Pr[A(x) = P(x)] \ge \frac{1}{2} + \frac{1}{q(n)},$$

for a fixed polynomial $q(n)$. Show that there is also an algorithm B that runs in polynomial time, and

$$\Pr[B(x) = P(x)] \geq 1 - 1/2^n.$$

(Hint: Repeatedly run A, use concentration inequalities)

Problem 4 (Concentration of MAX CUT)**.** Let G be an Erdös-Renyi random graph: for each pair (i, j), with probability $1/2$ it is an edge in G. All the edges are chosen independently. Let $c(G)$ be the capacity of the MAX-CUT in graph G. Show that

$$\Pr[|c(G) - \mathbb{E}[c(G)]| \geq \lambda\sqrt{\frac{n(n-1)}{2}}] \leq 2e^{-\lambda^2/2}.$$

(Hint: use bounded difference/McDiarmid's inequality)