COMPSCI630 Randomized Algorithms
Assignment 2

Due Date: Apr. 1, 2016 in class.

Problem 1 (Communications)**.** Alice has a unit vector $x \in \mathbb{R}^n$, Bob has a unit vector $y \in \mathbb{R}^n$. The two vectors either has an angle that is at most α, or at least β ($0 \leq \alpha < \beta \leq \pi$. They also share public bits of randomness.

Show that they can communicate $O(1/(\beta - \alpha)^2)$ bits, and determine whether the angle is at most α or at least β with success probability at least 3/4.

Note that communicating a real number (with reasonable precision) counts as $O(\log n)$ bits and is not allowed. (Hint: Use hyperplanes.)

Problem 2 (Graph sparsifier)**.** Let C_n be a cycle with n vertices. That is, the graph has n edges: there is an edge between $(i, i+1)$ for $i = 1, 2, ..., n-1$ and there is also an edge $(n, 1)$.

Let P_n be a path with n vertices. That is, the graph has $n-1$ edges, there is an edge between $(i, i+1)$ for $i = 1, 2, ..., n-1$.

(a) Let $w_C(S, \bar{S})$ be the capacity of the cut in C_n, and $w_P(S, \bar{S})$ be the capacity of the cut in P_n. Show that for every cut $(S, \bar{S})$,

$$w_P(S, \bar{S}) \leq w_C(S, \bar{S}) \leq 2w_P(S, \bar{S}).$$

(b) Let $\mathcal{L}_C$ be the Laplacian matrix of C, $\mathcal{L}_P$ be the Laplacian matrix of P, show that there exists a vector $x \in \mathbb{R}^n$ such that

$$x^\top \mathcal{L}_C x \geq n x^\top \mathcal{L}_P x.$$

(So, P_n is a cut sparsifier of C_n, but not a spectral sparsifier.)

Problem 3 (Johnson-Lindenstraus for matrix operations)**.** Suppose $A \in \mathbb{R}^{m \times r}$ and $B \in \mathbb{R}^{r \times n}$. We would like to approximately compute $C = AB$. Let $U \in \mathbb{R}^{r \times s}$ (think of $s \ll r$) be a random matrix whose entries are drawn independently from standard Gaussian distribution $N(0, 1)$. We will compute $\hat{C} = \frac{1}{s} AUU^\top B$.

(a) Show that $\hat{C}$ can be computed within time $O((m+n)rs + smn)$.

(b) Show that $\mathbb{E}[\hat{C}] = C$.

(c) Show that $\mathbb{E}[\|\hat{C} - C\|_F^2] \leq \frac{3\|A\|_F^2\|B\|_F^2}{s}$.
Hint: First bound variance when $s = 1$, then use independence between columns of U.
Some useful inequalities are

$$\langle u, v \rangle \leq \|u\|\|v\|, \quad a \cdot b \leq \frac{a^2 + b^2}{2}.$$