COMPSCI630 Randomized Algorithms
Assignment 3

Due Date: Apr. 15, 2016 in class.

Problem 1 (Unfair Bet)**.** Alice and Bob are playing a game. Alice has a special card: both sides are identical card backs. Bob has a regular Ace of Spades (see Figure below). In each round, they put the two cards into a bag, picks one card out and put it on the table. If the current side they see is not card back, then they restart the round. Otherwise they turn the card to check whether it is Alice's or Bob's. If it is Alice's card Alice gets 1 dollar from Bob, otherwise Bob gets 1 dollar from Alice.

Clarification: Every round will have a winner. If a restart happens it is still in the same round and the round goes on until a winner is decided.

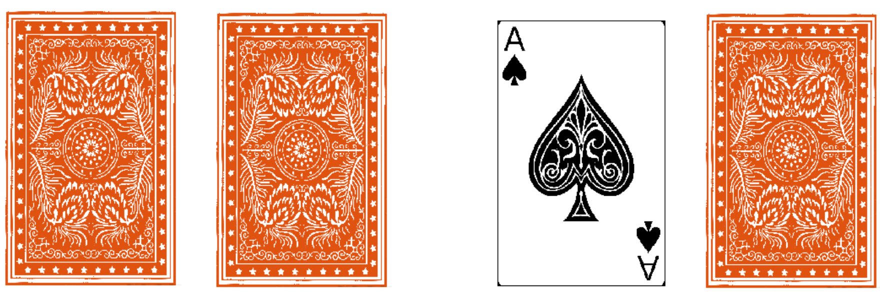

Figure 1: Left to Right: Alice's card (front, back), Bob's card (front, back)

(a) What is the probability that Alice wins a round? (We assume when they pick up the card and put it on the table, either side of either card has a equal probability of facing them.)

(b) Suppose both Alice and Bob have n dollars to begin with, and the game finishes when one of them has got all the $2n$ dollars. What is the probability of Alice winning the game? (Hint: Formalize the game as a Markov Chain, represent the Markov Chain using a weighted undirected graph, finally use resistor network/system of equations to compute the probability of winning.)

Problem 2 (Volume Estimation)**.** In this problem we are going to describe an algorithm for estimating the volume of complex object in high dimensions.

Suppose X is an object in $\mathbb{R}^d$. Given a point $x \in \mathbb{R}^d$, we can efficiently decide whether $x \in X$. Let $R_0, R_1, ..., R_n$ be n concentric spheres in d dimensions with the property that R_0 is completely inside X, and X is completely inside R_n. The radii of these spheres are $r_0 < r_1 < \cdots < r_n$ (center and radii of the spheres are known). See the Figure.

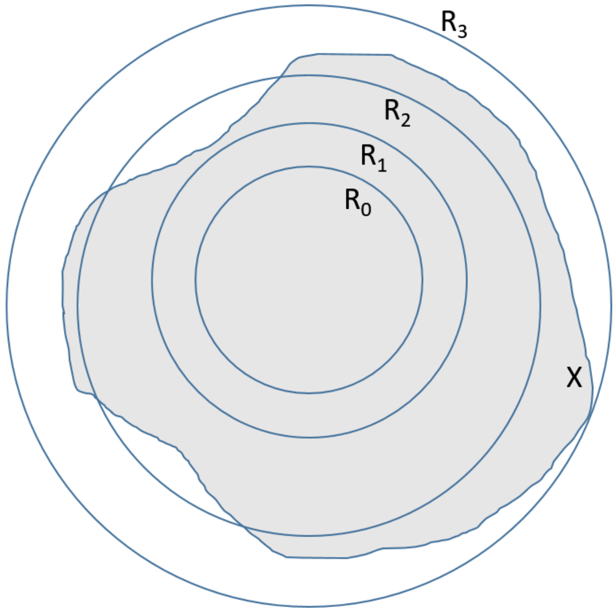

Figure 2: Object X and spheres $R_0, ..., R_3$.

Let S_i be the intersection of R_i and X, and let v_i be the volume of S_i. Clearly v_0 is the volume of R_0 which we can compute, and v_n is the volume of X. We will also assume $v_i/v_{i+1} \geq 0.5$ for all $i = 0, 1, ..., n-1$.

(a) Suppose we can sample points uniformly at random from S_{i+1}, show that with $O(\frac{\log(1/\eta)}{\epsilon^2})$ samples we can get an estimate q_i such that $|q_i - v_i/v_{i+1}| \leq \epsilon$ with probability at least $1 - \eta$ (for any $\eta > 0$).

(b) Now suppose the samples of S_{i+1} actually came from a fast mixing Markov Chain, that can achieve total variational difference β to the uniform distribution in S_{i+1} in time $O(\log 1/\beta)$. Show that using such a Markov Chain, we can get an estimate q_i that satisfies the same condition ($|q_i - v_i/v_{i+1}| \leq \epsilon$ with probability at least $1-\eta$) with running time $O(\frac{(\log(1/\eta))\log(1/\epsilon)}{\epsilon^2})$.

(Hint: Total variational difference being smaller than β means the probability of any event can differ by at most β.)

(c) Show that if $|q_i - v_i/v_{i+1}| \leq \epsilon$ for all $i = 0, 1, 2, ..., n-1$, let $\hat{v}_n = v_0 q_0 q_1 q_1 \cdots q_{n-1}$, then $\frac{|\hat{v}_n - v_n|}{v_n} \leq (1 + 2\epsilon)^n - 1$. (Note that the error is small if $\epsilon \ll 1/n$.)

(d) Show that if we can construct fast mixing Markov Chains as in (b), there is a polynomial time algorithm that can get an estimate $\hat{v}_n$ such that $\frac{1}{2}v_n \leq \hat{v}_n \leq \frac{3}{2}v_n$ with probability at least $1 - 1/n$.

(Hint: choose the right η and do union bound.)