Repeated Games and a Peak Beyond Nash Equilibrium

February 10, 2017

Catherine Moon
In systems of multiple self-interested agents, we cannot impose behavior on the agents.

Prisoner’s Dilemma, studied in “Dominance” of Vince’s Game Theory lecture.

<table>
<thead>
<tr>
<th></th>
<th>Confess</th>
<th>Don’t Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confess</td>
<td>-2, -2</td>
<td>0, -3</td>
</tr>
<tr>
<td>Don’t Confess</td>
<td>-3, 0</td>
<td>-1, -1</td>
</tr>
</tbody>
</table>

Cooperation is impossible in one-shot version of this game.
**Repeated Games?**

- In systems of multiple self-interested agents, we cannot impose behavior on the agents.
- Prisoner’s Dilemma, studied in “Dominance” of Vince’s Game Theory lecture.
- Finitely repeated game: unraveling through backwards induction – cooperation is impossible as well.

<table>
<thead>
<tr>
<th></th>
<th>confess</th>
<th>don’t confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>confess</td>
<td>-2, -2</td>
<td>0, -3</td>
</tr>
<tr>
<td>don’t confess</td>
<td>-3, 0</td>
<td>-1, -1</td>
</tr>
</tbody>
</table>
RepeateD Games?

• In systems of multiple self-interested agents, we cannot impose behavior on the agents

• Prisoner’s Dilemma, studied in “Dominance” of Vince’s Game Theory lecture

<table>
<thead>
<tr>
<th></th>
<th>confess</th>
<th>don’t confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>confess</td>
<td>-2, -2</td>
<td>0, -3</td>
</tr>
<tr>
<td>don’t confess</td>
<td>-3, 0</td>
<td>-1, -1</td>
</tr>
</tbody>
</table>

• Infinitely repeated: If I value future enough (discount factor $\delta$), then cooperative action may be sustained

• Folk Theorem
Maximal Cooperation in Repeated Games on Social Networks (IJCAI-15)

Catherine Moon and Vincent Conitzer
Duke University
RepeateD Game in Social Network

- **Common assumption**: an agent’s behavior is instantly observable to all other agents

- What if there is a delay in knowledge propagation due to network structure?

**Question**: Under what conditions can we still sustain cooperation, and can we compute the resulting equilibria?
**Motivating Example**

- Pollution reduction agreement

  ![Diagram](image)

  - Directionality
  - Cooperation involving cost and benefit

- Set of cooperating agents

  ![Diagram](image)

{Korea, Japan} or ∅
**MODEL**

- Set of agents $A$  
- Directed graph $G = (V, E)$  
- For all $i \in V$, cooperation cost $\kappa_i$  
- For all $(i, j) \in E$, cooperation benefit $\beta_{i,j}$  
- $x_{i,t} \in \{0,1\}$  
- Defection is irreversible  
- Agent $i$’s round $t$ payoff:  
  
  $$-x_{i,t} \kappa_i + \sum_{j:(j,i) \in E} x_{j,t} \beta_{j,i}$$

- Each period’s payoff is discounted by discount factor $\delta$
OUTLINE

• Analysis in Nash Equilibrium

• Algorithm for finding the unique maximal set of cooperating agents

• Experimental analysis on random graphs and observation of phase transition

• Extending equilibrium notions: Credibility of threats and Credible Equilibrium
DEFINITIONS

Let $S \subseteq A$ for a given network $G$

- Player $i$’s strategy **grim trigger for** $S$ consists of $i$ cooperating until some player $j \in S$ with $(j, i) \in E$ defects, which is followed by $i$ defecting (forever)

- The **grim trigger profile** $T[S]$ consists of all the agents in $S$ playing the grim trigger strategy for $S$, and all other players always defecting

- For any subset $S$ of the agents and any $i, j \in S$, the **distance from $i$ to $j$ through $S$**, denoted $d(i, j, S)$, is the length of the shortest path from $i$ to $j$ that uses only agents in $S$. For a set of agents $G \subseteq S$, $d(G, j, S) = \min_{i \in G} d(i, j, S)$
THEORETICAL ANALYSIS: EQUILIBRIUM

Let \( S \subseteq A \) for a given network \( G \)

**Proposition 1:** \( T[S] \) is an equilibrium if and only if \( \forall i \in S \)

\[
\sum_{j \in S : (j,i) \in E} \delta d(i,j,S) \beta_{j,i} \geq \kappa_i
\]

**Sketch:**

- Every player outside \( S \) is best-responding
- For \( i \in S \), cooperate if
  
  total utility loss from neighbors eventually defecting
  
  \[ \geq \]
  
  total utility gain from reduced effort
**Grim Trigger is WLOG**

**Proposition 2:** Suppose there exists a pure-strategy equilibrium in which $S$ is the set of players that cooperates forever. Then $T[S]$ is also an equilibrium.

**Sketch:**

- For a given equilibrium, consider some time period $\tau$ at which every player outside $S$ has defected (on the path of play).
- If player $i$ considers defecting at this point, total utility loss from neighbors eventually defecting is at most that from everyone in $S$ playing $T[S]$. 
MONOTONICITY

Let $S, S' \subseteq A$ for a given network $G$

Lemma 3: If $S \subseteq S'$ and the incentive constraint from Proposition 1 holds for $i$ relative to $S$, then it also holds for $i$ relative to $S'$.

Sketch:
• We argue that
  \[
  \sum_{j \in S': (j,i) \in E} \delta^{d(i,j,S')} \beta_{j,i} \geq \sum_{j \in S: (j,i) \in E} \delta^{d(i,j,S)} \beta_{j,i}
  \]
  1. All summands are nonnegative
  2. For any $i,j$, $d(i,j,S') \leq d(i,j,S)$
  3. Because $\delta < 1$, $\delta^{d(i,j,S')} \geq \delta^{d(i,j,S)}$
Maximality

Let \( S, S' \subseteq A \) for a given network \( G \)

**Proposition 4:** If \( T[S] \) and \( T[S'] \) are both equilibria, then so is \( T[S \cup S'] \)

**Sketch:**

- Consider some \( i \in S \cup S' \); WLOG, suppose \( i \in S \)
- Need to show: incentive constraint from Proposition 1 holds for \( i \) relative to \( S \cup S' \)
- Follows from Lemma 3, \( T[S] \) being an equilibrium, and \( S \subseteq S \cup S' \)
Algorithm 1

1: \( D_{\text{elimination}} \leftarrow \text{true} \)
2: while \( D_{\text{elimination}} = \text{true} \) do
3: \( L_{\text{defectors}} \leftarrow \emptyset \)
4: \( I \leftarrow \text{IncomingEdges}(E) \)
5: \( O \leftarrow \text{OutgoingEdges}(E) \)
6: for all \( i \in S^{\text{current}} \) do
7: \( L^i \leftarrow \text{ShortestPath}(i, j, E) \ ( \forall j \in S^{\text{current}} ) \)
8: \( C^i \leftarrow \text{IncentiveConstraint}(L^i, \kappa_i, \{ \beta_{j,i} \}_{j:(j,i) \in E}, \delta) \)
9: if \( C^i = \text{false} \) then
10: add \( i \) to \( L_{\text{defectors}} \)
11: end if
12: end for
13: for all \( i \in L_{\text{defectors}} \) do
14: remove \((i, j)\) from \( E \) for all \( j \)
15: remove \( i \) from \( S^{\text{current}} \)
16: end for
17: end while

IncentiveConstraint\((L^i, \kappa_i, \{ \beta_{j,i} \}_{j:(j,i) \in E}, \delta)\) checks the incentive constraint inequality in Proposition 1. If it is satisfied, it returns true, indicating \( i \)'s willingness to cooperate.
Experimental Analysis

Additional assumptions:

• For all $i \in S$, $\kappa_i = \sum_{i \in S: (i,j) \in E} 1$
• For all $(j, i) \in E$, $\beta_{j,i} = \beta$

Random graph models:

• Erdős–Rényi (ER)
• Barabási–Albert preferential-attachment (PA)
FINE LINE BETWEEN COMPLETE COOPERATION AND DEFECTION
Agents’ defection probabilities for different values of $\beta$ and $\delta$
**Simulation and Phase Transition**

Average number of iterations needed until convergence for different values of $\beta$ and $\delta$
CREDIBILITY OF THREATS IN EQUILIBRIUM

• The maximal set of cooperating agents, $S^*$, in Nash Equilibrium may involve threats of grim-trigger defection that are not credible.
CREDIBLE EQUILIBRIUM

• Equilibrium extensions beyond NE:
  Subgame-perfect? Perfect Bayesian? Sequential?

• Credible equilibrium: if an agent learns that some deviation from the equilibrium has taken place, then she will be best off following her equilibrium strategy regardless of her beliefs about which deviations have taken place, assuming others also follow their equilibrium strategies from this round on
CREDIBLE EQUILIBRIUM

**Lemma 7:** Sufficient to check singleton deviations

**Lemma 8:** Sufficient to check one-round postponement of punishment

**Theorem 9:** Suppose that for some set $S \subseteq A$, $T[S]$ is a Nash equilibrium. Then $T[S]$ is a credible equilibrium if and only if for any $k, i \in S$ with $(k, i) \in E$, it holds that

$$\kappa_i - \delta^{d(k,j,S)} \sum_{j \in D} \beta_{ji} \geq 0,$$

where

$$D = \{j \in I: d(k,j,S) + 1 \leq d(k,j,S_{-i})\}$$
CREDIBILITY OF MAXIMAL EQ

Top: Gradient graph showing agents’ defection probabilities; Bottom: gradient graph showing the fraction of cases where T[S*] is a CE
Thank You!