Robust Mechanism Design with Correlated Distributions

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Prior-Dependent Mechanisms

• In many situations we’ve seen, optimal mechanisms are prior dependent
  • Myerson auction for independent bidder valuations
  • Revenue maximization with efficient allocation with correlated bidder valuations (Cremer and McLean 1985; Albert, Conitzer, and Lopomo 2016)
  • Strong budget balanced mechanisms with correlated valuations (Kosenok and Severinov 2008)

• There are different degrees of prior dependence
• What if we don’t know the prior? Or can only estimate it using past reports in an auction?
What if the distribution isn’t well known?

![Graph showing relative revenue vs. number of samples for Ex-Post and Bayesian methods.](image-url)
Painting LP Example

• We make reproductions of two paintings

\[
\begin{align*}
\text{maximize} \quad & 3x + 2y \\
\text{subject to} \quad & 4x + 2y \leq 16 \\
& x + 2y \leq 8 \\
& x + y \leq 5 \\
& x \geq 0 \\
& y \geq 0
\end{align*}
\]

• Painting 1 sells for $30, painting 2 sells for $20

• Painting 1 requires 4 units of blue, 1 green, 1 red

• Painting 2 requires 2 blue, 2 green, 1 red

• We have 16 units blue, 8 green, 5 red
Mis-Estimation of the Objective

\[
\begin{align*}
\text{maximize} & \quad 3x + 2y \\
\text{subject to} & \quad 4x + 2y \leq 16 \\
& \quad x + 2y \leq 8 \\
& \quad x + y \leq 5 \\
& \quad x \geq 0 \\
& \quad y \geq 0
\end{align*}
\]

Optimal solution: \(x=3, y=2\)

Estimated solution: \(x=2, y=3\)

Objective Value with Optimal Solution: 13

Objective Value with Estimated Solution: 12
Mis-Estimation of the Constraints

\[
\begin{align*}
\text{maximize} & \quad 3x + 2y \\
\text{subject to} & \quad 4x + 2y \leq 16 \\
& \quad x + 2y \leq 8 \\
& \quad x + y \leq 5 \\
& \quad x \geq 0 \\
& \quad y \geq 0
\end{align*}
\]

optimal solution: \( x=3, y=2 \)
Problem Description

- A monopolistic seller with one item
- A single bidder with type $\theta \in \Theta$ and valuation $v(\theta)$
- An external signal $\omega \in \Omega$ and distribution $\pi(\theta, \omega)$
The Mechanism Design Problem

• Objective for mechanism design with optimal revenue

\[
\max_{p(\theta, \omega), x(\theta, \omega)} \sum_{(\theta, \omega) \in \Theta \times \Omega} \pi(\theta, \omega) x(\theta, \omega)
\]

• Ex-Post Individual Rationality, for all \( \theta \in \Theta \) and \( \omega \in \Omega \):

\[
p(\theta, \omega) v(\theta) - x(\theta, \omega) \geq 0
\]

• Ex-Interim Individual Rationality, for all \( \theta \in \Theta \):

\[
\sum_{\omega} \pi(\omega|\theta)(p(\theta, \omega)v(\theta) - x(\theta, \omega)) \geq 0
\]

• Ex-Post Incentive Compatibility, for all \( \theta \in \Theta \) and \( \omega \in \Omega \):

\[
p(\theta, \omega)v(\theta) - x(\theta, \omega) \geq p(\theta', \omega)v(\theta) - x(\theta', \omega)
\]

• BNE Incentive Compatibility, for all \( \theta \in \Theta \):

\[
\sum_{\omega} \pi(\omega|\theta)(p(\theta, \omega)v(\theta) - x(\theta, \omega)) \\
\geq \sum_{\omega} \pi(\omega|\theta)(p(\theta', \omega)v(\theta) - x(\theta', \omega))
\]
Consistent Set of Distributions

Let $P(A)$ be the set of probability distributions over the set $A$. Then the space of all probability distributions over $\Theta \times \Omega$ can be represented as $P(\Theta \times \Omega)$. A subset $\mathcal{P}(\hat{\pi}) \subseteq P(\Theta \times \Omega)$ is a consistent set of distributions for the estimated distribution $\hat{\pi}$ if the true distribution, $\pi$, is guaranteed to be in $\mathcal{P}(\hat{\pi})$ and $\pi \in \mathcal{P}(\hat{\pi})$. 
Something Between Ex-Post and Bayesian

• We need something between Ex-Post/Dominant Strategy and Interim/BNE.

• We need a prior dependent notion of incentive compatibility and individual rationality that is robust to small mis-estimations of the distributions.

• There is a catch: we can now allow the bidder to report the true distribution at the same time as he reports his valuation (Revelation Principle)
  • A mechanism is a probability of allocation of the item, $p(\theta, \pi, \omega)$, and a payment, $x(\theta, \pi, \omega)$, that depends on the reported type, $\theta$, the reported distribution, $\pi$, and the external signal $\omega$.
  • Define the bidder’s expected utility from reporting $(\theta', \pi')$ when his true valuation is $(\theta, \pi)$ and the external signal is $\omega$ as:

$$U(\theta, \pi, \theta', \pi', \omega) = p(\theta', \pi', \omega)v(\theta) - x(\theta', \pi', \omega)$$
Robust Individual Rationality and Incentive Compatibility

A mechanism is *robust individually rational* for estimated bidder distribution \( \hat{\pi} \) and consistent set of distributions \( \mathcal{P}(\hat{\pi}) \) if for all \( \theta \in \Theta \) and \( \pi \in \mathcal{P}(\hat{\pi}) \):

\[
\sum_{\omega \in \Omega} \pi(\omega | \theta) U(\theta, \pi, \theta, \pi, \omega) \geq 0
\]

A mechanism is *robust incentive compatible* for estimated bidder distribution \( \hat{\pi} \) and consistent set of distributions \( \mathcal{P}(\hat{\pi}) \) if for all \( \theta, \theta' \in \Theta \) and \( \pi, \pi' \in \mathcal{P}(\hat{\pi}) \):

\[
\sum_{\omega \in \Omega} \pi(\omega | \theta) U(\theta, \pi, \theta, \pi, \omega) \geq \sum_{\omega \in \Omega} \pi(\omega | \theta') U(\theta, \pi, \theta', \pi', \omega)
\]
Should we listen to the bidder’s reported belief?

- The revelation principle still holds, and we can always construct a mechanism where the bidder reports his belief $\pi(\omega|\theta)$ along with his type truthfully.
- We will only consider, with loss of generality, mechanisms that do not take the reported valuation, $\pi$, into account.
  - A mechanism is a probability of allocation of the item, $p(\theta, \omega)$, and a payment, $x(\theta, \omega)$, that depends on the reported type, $\theta$, and the external signal, $\omega$.
  - Define the bidder’s expected utility from reporting $(\theta', \pi')$ when his true valuation and distribution is $(\theta, \pi)$ and the external signal is $\omega$ as:
    \[
    U(\theta, \pi, \theta', \omega) = p(\theta', \omega)v(\theta) - x(\theta', \omega)
    \]
- Note that this is not without loss of generality, but it reduces the set of payments and probabilities that must be specified to a finite set.
  - Once we have a finite set, we can use techniques from automated mechanism design to design the optimal restricted robust mechanism.
Linear Program for Optimal Restricted Robust Mechanism

\[ \max_{p(\theta, \omega), x(\theta, \omega)} \sum_{(\theta, \omega) \in \Theta \times \Omega} \hat{\pi}(\theta, \omega)x(\theta, \omega) \]

Subject to:

\[ \sum_{\omega \in \Omega} \pi(\omega | \theta)U(\theta, \pi, \theta, \omega) \geq \sum_{\omega \in \Omega} \pi(\omega | \theta)U(\theta, \pi, \theta', \omega) \quad \forall \theta, \theta' \in \Theta \]

and \( \forall \pi \in P(\hat{\pi}) \)

\[ \sum_{\omega \in \Omega} \pi(\omega | \theta)U(\theta, \pi, \theta, \omega) \geq 0 \quad \forall \theta \in \Theta \text{ and } \forall \pi \in P(\hat{\pi}) \]

\[ 0 \leq p(\theta, \omega) \leq 1 \quad \forall \theta \in \Theta \text{ and } \forall \omega \in \Omega \]

We have an infinite number of constraints!
Constraint Generation Linear Program

(IR) For all $\theta \in \Theta$: \( \min_{\pi} \sum_{\omega \in \Omega} \pi(\omega | \theta) U(\theta, \pi, \theta, \omega) \)

(IC) For all $\theta, \theta' \in \Theta$: \( \min_{\pi} \sum_{\omega \in \Omega} \pi(\omega | \theta)(U(\theta, \pi, \theta, \omega) - U(\theta, \pi, \theta', \omega)) \)

Subject to: \( \pi \in \mathcal{P}(\hat{\pi}) \)

• Solve the first linear program with a consistent set $\mathcal{P}'(\hat{\pi}) = \{\hat{\pi}\}$.
• Solve each of the linear programs, and then add the solution, $\pi$, to $\mathcal{P}'(\hat{\pi})$ if the objective value is less than 0.
• Re-solve the original linear program and repeat.
• Guaranteed to terminate in a polynomial number of iterations (Kozlov, Tarasov, and Khachiyan 1980)
• What is the issue with this set of linear programs?
  • We need the constraints to be finite!
Polyhedral Consistent Set

If the set $\mathcal{P}(\hat{\pi})$ can be characterized as an $n$-polyhedron, where $n$ is polynomial in the number of bidder and external signal types, then the optimal robust mechanism can be computed in polynomial time.

• Example: Suppose that we restrict our attention to consistent sets such that for all $\theta \in \Theta$ and $\omega \in \Omega$, $\pi(\omega|\theta) \in [\underline{\pi}(\omega|\theta), \overline{\pi}(\omega|\theta)]$. Then the constraints become:

$$\underline{\pi}(\omega|\theta) \leq \pi(\omega|\theta) \leq \overline{\pi}(\omega|\theta) \quad \forall \omega \in \Omega$$

$$\sum_{\omega \in \Omega} \pi(\omega|\theta) = 1$$
Robust spans the distance between Ex-Post and Bayesian

- Robust incentive compatibility and individual rationality contain ex-post and Bayesian IC and IR as special cases

- Suppose that the set $\mathcal{P}(\hat{\pi})$ is a singleton, i.e. $\mathcal{P}(\hat{\pi}) = \{\hat{\pi}\}$. Then robust IR becomes ex-interim IR:

$$\sum_{\omega \in \Omega} \hat{\pi}(\omega|\theta) U(\theta, \hat{\pi}, \theta, \omega) \geq 0 \quad \forall \theta \in \Theta$$

- Instead suppose that the consistent set is such that for all $\theta \in \Theta$ and $\omega \in \Omega$, $\pi(\omega|\theta) \in [0,1]$. Then the set of ex-post IR constraints appears in the robust IR set:

$$U(\theta, \hat{\pi}, \theta, \omega) \geq 0 \quad \forall \theta \in \Theta \text{ and } \omega \in \Omega$$

- Therefore, robust IR and IC spans the traditional set of constraints.
Is robust enough?

- Any reasonable estimation procedure will return a consistent set, $\mathcal{P}(\hat{\pi})$, as the entire set of possible distributions, $P(\Theta \times \Omega)$, i.e. $\mathcal{P}(\hat{\pi}) = P(\Theta \times \Omega)$.
  - This is because the definition guarantees that the true distribution, $\pi$, is in the consistent set.
- We really want to say that the true distribution is in the consistent set with high probability:

Let $P(A)$ be the set of probability distributions over the set $A$. Then the space of all probability distributions over $\Theta \times \Omega$ can be represented as $P(\Theta \times \Omega)$. A subset $\mathcal{P}_\epsilon (\hat{\pi}) \subseteq P(\Theta \times \Omega)$ is an $\epsilon$-consistent set of distributions for the estimated distribution $\hat{\pi}$ if the true distribution, $\pi$, is in $\mathcal{P}_\epsilon (\hat{\pi})$ with probability $1 - \epsilon$ and $\hat{\pi} \in \mathcal{P}_\epsilon (\hat{\pi})$. 
Finding the Consistent Set

• This is all well and good, but how do we actually find the consistent set if we just observe a bunch of samples from the distribution?

• The best way that I’ve found is to use Bayesian techniques
  • A bivariate discrete distribution can generally be modeled as a categorical distribution (every possible combination of reports is a category)
  • You can learn a categorical distribution using a Dirichlet distribution (it is the conjugate prior of the categorical distribution)
  • Start with a Dirichlet distribution with an uninformative prior \( \alpha = 1 \), then increment the element of \( \alpha \) by 1 every time that element is seen, i.e. if you see sample \((\theta', \omega')\) then \( \alpha((\theta', \omega')) = \alpha((\theta', \omega')) + 1 \)
  • The posterior is another Dirichlet distribution parameterized by the new \( \alpha \)

• Once the posterior Dirichlet distribution is calculated, you can sample from the distribution to get empirical confidence intervals

• Note that this can be done at the level of conditional distributions instead of the full distribution.
What happens in practice?

• True distribution is a discretized bivariate normal distribution
• Sample from true distribution $N$ times
• Estimate the distribution using the previously described Bayesian procedure
• Calculate empirical confidence intervals for each element of the conditional distribution as an interval with odds of being outside of the interval as $\frac{\epsilon}{|\Omega|}$, i.e. $\pi(\omega|\theta) \in \left[\pi(\omega|\theta), \pi(\omega|\theta)\right]$ with probability $1 - \frac{\epsilon}{|\Omega|}$.
  • This is because we are need the entire distribution to be inside of the interval
• Parameters unless otherwise specified
  • Correlation $= .5$
  • $\epsilon = .05$
  • $\Theta = \{1,2, ..., 10\}$, $\nu(\theta) = \theta$
  • $\Omega = \{1,2, ..., 10\}$
Open questions in robust mechanism design

• How do we optimally compute the $\varepsilon$-consistent set?
• Can we bound the potential loss due to the lock of guaranteed IC and IR for $\varepsilon$-robust mechanisms?
  • If we can bound the maximum payment, this should be possible
• Can we use a prior over beliefs (the Dirichlet distribution we compute, for example) to construct a mechanism that depends on the reported belief of the bidder?
• What is the optimal approach to binning?
• What is the sample complexity of an $\varepsilon$-approximation to the optimal mechanism?
  • This will depend on characteristics of the underlying distribution (Albert, Conitzer, and Stone 2017 (AAMAS))
  • Can we characterize the sample complexity in terms of the separation between points in belief space?
• Are there simpler robust mechanisms that scale more easily to more bidders?
How do we learn online?

• Everything we’ve discussed assumes that someone hands us a bunch of samples.

• In reality, we will need to elicit these reports from bidders in multiple rounds, but this is not an easy task

• If bidder’s are myopic (i.e., they don’t lie to mislead future auctions)
  • Then if the mechanism fails to be IR, we won’t see a report for that bidder type at all, skewing the distribution
  • If the mechanism fails to be IC, we will see a wrong report, skewing the distribution
  • Can use an ex-post mechanism for early rounds, and then stop learning. What is the regret?

• If bidders are strategic
  • They may lie even when the mechanism is ex-post in order to influence future rounds
  • Must give strict incentives to tell the truth in early rounds