Securities & Expressive Securities Markets

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Call options

- A (European) call option $C(S, k, t)$ gives you the right to buy stock $S$ at (strike) price $k$ on (expiry) date $t$
  - American call option can be exercised early
  - European one easier to analyze

- How much is a call option worth at time $t$ (as a function of the price of the stock)?

\[ p_t(C) \]

Graph:
- Vertical axis: value
- Horizontal axis: $p_t(S)$, $k$
Put options

- A (European) put option $P(S, k, t)$ gives you the right to sell stock $S$ at (strike) price $k$ on (expiry) date $t$.
- How much is a put option worth at time $t$ (as a function of the price of the stock)?

![Graph showing the value of a put option $p_t(P)$ as a function of the price of the stock $p_t(S)$ and the strike price $k$. The value of the put option decreases as the price of the stock increases.]
Bonds

- A bond $B(k, t)$ pays off $k$ at time $t$
Stocks

value

$k$

$p_t(S)$

$p_t(S)$
Selling a stock (short)

\[
\text{value} \uparrow
\]

\[
k
\]

\[
p_t(S)
\]

\[
-p_t(S)
\]
A portfolio

- One call option $C(S, k, t)$ + one bond $B(k, t)$
Another portfolio

- One put option $P(S, k, t)$ + one stock $S$

same thing!
Put-call parity

- \( C(S, k, t) + B(k, t) \) will have the same value at time \( t \) as \( P(S, k, t) + S \) (regardless of the value of \( S \))
- **Assume** stocks pay no dividends
- Then, portfolio should have the same value at any time before \( t \) as well
- I.e., for any \( t' < t \), it should be that \( p_{t'}(C(S, k, t)) + p_{t'}(B(k, t)) = p_{t'}(P(S, k, t)) + p_{t'}(S) \)

- **Arbitrage argument**: suppose (say) \( p_{t'}(C(S, k, t)) + p_{t'}(B(k, t)) < p_{t'}(P(S, k, t)) + p_{t'}(S) \)
- Then: buy \( C(S, k, t) + B(k, t) \), sell (short) \( P(S, k, t) + S \)
- Value of portfolio at time \( t \) is 0
- Guaranteed profit!
Another perspective: auctioneer

- **Auctioneer** receives buy and sell offers, has to choose which to accept
- E.g.: offers received: buy(S, $10); sell(S, $9)
- Auctioneer can accept both offers, profit of $1
- E.g. (put-call parity):
  - sell(C(S, k, t), $3)
  - sell(B(k, t), $4)
  - buy(P(S, k, t), $5)
  - buy(S, $4)
- Can accept all offers at no risk!
“Butterfly” portfolio

- 1 call at strike price $k-c$
- -2 calls at strike $k$
- 1 call at strike $k+c$

\[ p_t(S) \]

\[ \text{value} \]

\[ c \]
Another portfolio

- Can we create this portfolio?
Yet another portfolio

• How about this one?
Two different stocks

- A portfolio with $C(S_1, k, t)$ and $S_2$
Another portfolio

• Can we create this portfolio? (In effect, a call option on $S_1 + S_2$)
A useful property

- Suppose your portfolio pays off $f(p_t(S_1), p_t(S_2)) = f_1(p_t(S_1)) + f_2(p_t(S_2))$ (additive decomposition over stocks).

- This is all we know how to do.

- Then: $f(x_1, x_2) - f(x_1, x_2') = f(x_1) + f(x_2) - f(x_1) - f(x_2') = f(x_2) - f(x_2') = f(x_1', x_2) - f(x_1', x_2')$
• Can we create this portfolio?  
(In effect, a call option on S₁+S₂)
Securities conditioned on finite set of outcomes

- E.g., InTrade: security that pays off 1 if Trump is the Republican nominee in 2016
- Can we construct a portfolio that pays off 1 if Clinton is the Democratic nominee AND Trump is the Republican nominee?

<table>
<thead>
<tr>
<th></th>
<th>Trump not nom.</th>
<th>Trump nom.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clinton not nom.</td>
<td>$0</td>
<td>$0</td>
</tr>
<tr>
<td>Clinton nom.</td>
<td>$0</td>
<td>$1</td>
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</tbody>
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Arrow-Debreu securities

• Suppose S is the set of all states that the world can be in tomorrow
• For each s in S, there is a corresponding Arrow-Debreu security that pays off 1 if s happens, 0 otherwise
• E.g., s could be: Clinton is nominee and Trump is nominee and $S_1$ is at $4$ and $S_2$ at $5$ and butterfly 432123 flaps its wings in Peru and…
• Not practical, but conceptually useful
• Can think about Arrow-Debreu securities within a domain (e.g., states only involve stock trading prices)
• Practical for small number of states
With Arrow-Debreu securities you can do anything…

• Suppose you want to receive $6 in state 1, $8 in state 2, $25 in state 3

• … simply buy 6 AD securities for state 1, 8 for state 2, 25 for state 3

• Linear algebra: Arrow-Debreu securities are a basis for the space of all possible securities
The auctioneer problem

• Tomorrow there must be one of ☀️ ☂️ ☣️

• Agent 1 offers $5 for a security that pays off $10 if ☂️ or ☣️

• Agent 2 offers $8 for a security that pays off $10 if ☀️ or ☣️

• Agent 3 offers $6 for a security that pays off $10 if ☀️

• Can we accept some of these at offers at no risk?
Reducing auctioneer problem to ~combinatorial exchange winner determination problem

• Let \((x, y, z)\) denote payout under ☀️, ⛈️, ♨️, respectively

• Previous problem’s bids:
  – 5 for \((0, 10, 10)\)
  – 8 for \((10, 0, 10)\)
  – 6 for \((10, 0, 0)\)

• Equivalently:
  – \((-5, 5, 5)\)
  – \((2, -8, 2)\)
  – \((4, -6, -6)\)

• Sum of accepted bids should be \((\leq 0, \leq 0, \leq 0)\) to have no risk

• Sometimes possible to partially accept bids
A bigger instance (4 states)

- Objective: maximize our **worst-case** profit
- 3 for (0, 0, 11, 0)
- 4 for (0, 2, 0, 8)
- 5 for (9, 9, 0, 0)
- 3 for (6, 0, 0, 6)
- 1 for (0, 0, 0, 10)

- What if they are partially acceptable?
Settings with large state spaces

• Large = exponentially large
  – Too many to write down
• Examples:
• \( S = S_1 \times S_2 \times \ldots \times S_n \)
  – E.g., \( S_1 = \{\text{Clinton not nom.}, \text{Clinton nom.}\} \), \( S_2 = \{\text{Trump not nom.}, \text{Trump nom.}\} \), \( S = \{(-C, -T), (-C, +T), (+C, -T), (+C, +T)\} \)
  – If all \( S_i \) have the same size \( k \), there are \( k^n \) different states
• \( S \) is the set of all rankings of \( n \) candidates
  – E.g., outcomes of a horse race
  – \( n! \) different states (assuming no ties)
Bidding languages

• How should trader (bidder) express preferences?
• Logical bidding languages [Fortnow et al. 2004]:
  – (1) “If Trump nominated OR (Cruz nominated AND Clinton nominated), I want to receive $10; I’m willing to pay $6 for this.”
• If the state is a ranking [Chen et al. 2007] :
  – (2a) “If horse A ranks 2\textsuperscript{nd}, 3\textsuperscript{rd}, or 4\textsuperscript{th} I want to receive $10; I’m willing to pay $6 for this.”
  – (2b) “If one of horses A, C, D rank 2\textsuperscript{nd}, I want to receive $10; I’m willing to pay $6 for this.”
  – (2c) “If horse A ranks ahead of horse C, I want to receive $10; I’m willing to pay $6 for this.”
• Winner determination problem is NP-hard for all of these, except for (2a) and (2b) which are in P if bids can be partially accepted
A different computational problem closely related to (separation problem for) winner determination

• Given that the auctioneer has accepted some bids, what is the worst-case outcome (state) for the auctioneer?
• For example:
  • Must pay 2 to trader A if horse X or Z is first
  • Must pay 3 to trader B if horse Y is first or second
  • Must pay 6 to trader C if horse Z is second or third
  • Must pay 5 to trader D if horse X or Y is third
  • Must pay 1 to trader E if horse X or Z is second
Reduction to weighted bipartite matching

- Must pay 2 to trader A if horse X or Z is first
- Must pay 3 to trader B if horse Y is first or second
- Must pay 6 to trader C if horse Z is second or third
- Must pay 5 to trader D if horse X or Y is third
- Must pay 1 to trader E if horse X or Z is second