Due on April 22, 2019
107 points total

General directions: We will exclusively use Python 3 for our programming assignments, and allow only the use of modules in the Python 3 standard library unless explicitly specified otherwise on an individual assignment basis. This forbids the use of common third-party libraries such as Numpy, Sympy, etc., but not the use of math or io.

Unless specified otherwise, for the X-th homework, download the single "hwX_skeleton.py" file from the course website, and rename it to "hwX.py" on your machine. When you are done and ready to submit, upload your file named **exactly** "hwX.py" on Gradescope for assignment "HW X (Programming)." When you upload your file, the autograder will run a simple test for each function so that you can confirm it was properly uploaded and executed. Generally, if an assignment involves printing or writing a file in a specific format, there will be at least one simple test that checks your output is formatted as we expect. These tests are not worth any credit — once the due date is over, your submission will be graded by a collection of additional test cases.

All answers to non-programming questions must be typed, preferably using LaTeX. If you are unfamiliar with LaTeX, you are strongly encouraged to learn it. However, answers typed in other text processing software and properly converted to a PDF file will also be accepted. To submit your file, upload your PDF on Gradescope for assignment "HW X (PDF)." Handwritten answers or PDF files that cannot be opened will not be graded and will not receive any credit.

Finally, please read the detailed collaboration policy given on the course website. You are **not** allowed to discuss homework problems in groups of more than 3 students. **Failure to adhere to these guidelines will be promptly reported to the relevant authority without exception.**

Point values: Every problem has a specified amount of points which are awarded for the correctness of your solutions. In addition, each proof-oriented problem has an additional **style point**. In the homework handout, this is signified by a "+1" in the point value. To earn this point, your solutions should be clear, well organized, and easy to follow. This is to encourage not only perfectly correct solutions, but well presented ones.

Problem 1 (15+1 points)
Let n be an even positive integer, and let us construct a "bullseye" sequence of n squares $(S_1, \dots, S_n)$ as follows: S_1 is a square with side length 1, and the interior of S_1 is red. Now for every $i \geq 2$, the square S_i has side length i, and S_{i-1} is entirely contained in S_i. Every square is centered about the same point, and their orientations are identical.

The regions $S_i \setminus S_{i-1}$ alternate in color between blue and red. Thus, the border surrounding S_1 (i.e., the region $S_2 \setminus S_1$) is blue, the border surrounding S_2 (i.e., $S_3 \setminus S_2$) is red, and so on. Since n is even, the outermost border is always blue. See Fig. 1 for the picture when $n = 4$.

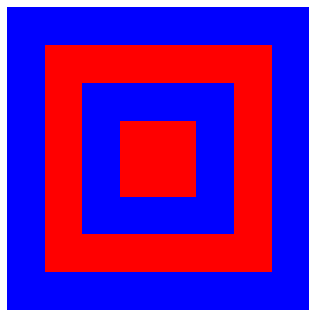

Figure 1: The figure corresponding to Problem 1 when $n = 4$.

Let R denote the total area of the regions colored red, and let B denote the total area of regions colored blue. Give a closed form expression (i.e., no summations) for $B/(R + B)$, and prove your result.

Problem 2 (15+1 points)
Consider a sequence of real numbers $(a_1, a_2, \dots)$ defined as follows:

- The first two terms are $a_1 = 1$ and $a_2 = 0.1$.
- If $n \geq 3$, then $a_n = 0.01 \cdot a_{n-2} + 0.1 \cdot a_{n-1}$.

Let S denote the sum of all of the terms of the sequence, that is, $S = \sum_{i=1}^{\infty} a_i$.

(a) (10 points) Prove that $S \leq 2$. (**Hint:** *How does the value of a_n compare to 2^{-n}?*)

(b) (5 points) State the exact value of S. (Justification is not required, but encouraged.)

Problem 3 (10+1 points)
Suppose there are $n \geq 3$ people who each simultaneously flip a coin that returns H with probability p and T with probability $1-p$. Compute the probability that there exists a person whose result is unique among all results, and justify your result.

Problem 4 (15+1 points)
Suppose we flip a fair coin n times, where n is an even positive integer. For every positive integer ℓ, an *ℓ-streak* is a consecutive sequence of ℓ flips that all return the same result. What is the probability of obtaining an $(n/2)$-streak? Justify your result.

Problem 5 (20+1 points)
Suppose a car engine manufacturer creates a faulty engine with probability 0.005. Furthermore, a faulty engine emits an odor with probability 0.95, while a non-faulty engine emits an odor with probability 0.1. Now let X be a fixed engine, and compute the following values. Every response should be justified.

(a) (5 points) What is the probability that X emits an odor?

(b) (5 points) Given that X emits an odor, what is the probability that X is faulty?

(c) (5 points) Given that X does not emit an odor, what is the probability that X is not faulty?

(d) (5 points) Suppose a novice mechanic concludes that an engine is faulty if and only if it emits an odor. The mechanic then examines X for an odor and arrives at a conclusion. What is the probability that the mechanic's conclusion is incorrect?

Problem 6 (25+2 points)
In this problem, we will study the notion of spanning trees in multigraphs. A *multigraph* is a graph in which multiple edges between two vertices is permitted. More formally, a *multiset* is a set in which repetition of elements is allowed (e.g., $\{1, 3, 1, 2\}$). A *multigraph* is a pair (V, E) where V is a non-empty finite set and E is a multiset containing subsets of V of size 2.

Now let $G = (V, E)$ be a multigraph, and let $e = \{u, v\}$ be an edge of G. We can *contract* the edge e to create a graph, denoted $G \setminus e$, as follows: the vertex set of $G \setminus e$ is equal to $V \setminus \{v\}$. We construct the edge set of $G \setminus e$ as follows:

1. Add all of the edges of G that do not contain v as an endpoint.
2. For every edge $\{v, x\}$ of G, add the edge $\{u, x\}$.

In other words, the vertices u and v become a single vertex that is adjacent to all vertices that were adjacent to u or v (or both). Furthermore, we let $G - e$ denote the graph G with e removed from the edge set. Now let $\tau(G)$ denote the number of spanning trees in G (if $|V| = 1$, then $\tau(G) = 1$, and if G is disconnected, then $\tau(G) = 0$).

(a) (15+1 points) Prove that for every $e \in E$, $\tau(G) = \tau(G - e) + \tau(G \setminus e)$.

(b) (10+1 points) Let e_1 and e_2 be two distinct edges of G, and let T be a spanning tree of G chosen uniformly at random (from the set of all spanning trees of G). What is the probability that T contains e_1 or e_2, but not both? Write your answer in terms of the τ function, and prove your result.