Due on February 25, 2019
137 points total

General directions: We will exclusively use Python 3 for our programming assignments, and allow only the use of modules in the Python 3 standard library unless explicitly specified otherwise on an individual assignment basis. This forbids the use of common third-party libraries such as Numpy, Sympy, etc., but not the use of math or io.

Unless specified otherwise, for the X-th homework, download the single "hwX_skeleton.py" file from the course website, and rename it to "hwX.py" on your machine. When you are done and ready to submit, upload your file named **exactly** "hwX.py" on Gradescope for assignment "HW X (Programming)." When you upload your file, the autograder will run a simple test for each function so that you can confirm it was properly uploaded and executed. Generally, if an assignment involves printing or writing a file in a specific format, there will be at least one simple test that checks your output is formatted as we expect. These tests are not worth any credit — once the due date is over, your submission will be graded by a collection of additional test cases.

All answers to non-programming questions must be typed, preferably using LaTeX. If you are unfamiliar with LaTeX, you are strongly encouraged to learn it. However, answers typed in other text processing software and properly converted to a PDF file will also be accepted. To submit your file, upload your PDF on Gradescope for assignment "HW X (PDF)." Handwritten answers or PDF files that cannot be opened will not be graded and will not receive any credit.

Finally, please read the detailed collaboration policy given on the course website. You are **not** allowed to discuss homework problems in groups of more than 3 students. **Failure to adhere to these guidelines will be promptly reported to the relevant authority without exception.**

Point values: Every problem has a specified amount of points which are awarded for the correctness of your solutions. In addition, each proof-oriented problem has an additional **style point**. In the homework handout, this is signified by a "+1" in the point value. To earn this point, your solutions should be clear, well organized, and easy to follow. This is to encourage not only perfectly correct solutions, but well presented ones.

Problem 1 (27+1 points)
For each relation on $\mathbb{R}$ below, state if the relation is reflexive, irreflexive, transitive, symmetric, anti-symmetric, and/or asymmetric. Proofs are only required for parts (d), (e), and (f).

(Recall that if R_1 and R_2 are relations, then $R_2 \circ R_1$ is a relation that contains (x, z) iff there exists y such that xR_1y and yR_2z. Also, if A and B are sets, then $A \oplus B$ is a set containing x iff x is in exactly one of A, B.)

For each symbol $*$ in $\{<, \leq, >, \geq, =, \neq\}$, we let R_* denote the relation $\{(x, y) : x * y\}$. For example, $R_< = \{(x, y) : x < y\}$.

(a) (3 points) $R_<$.

(b) (3 points) $R_< \oplus R_=$.

(c) (3 points) $R_\neq \circ R_\neq$.

(d) (6 points) $R_\leq \circ R_>$.

(e) (6 points) $R_> \setminus R_\neq$.

(f) (6 points) $\{(x, y) : x - y \in \mathbb{Q}\}$.

Problem 2 (30 points)
Recall that if R is a relation on a set A, then there are six properties of R that we often consider: reflexivity, irreflexivity, transitivity, symmetry, anti-symmetry, asymmetry. Implement the six corresponding functions that test for these properties in your "hw4.py" file; the skeleton file can be found on the course website as usual. The details of this problem are again found in the skeleton code.

Problem 3 (18+1 points)
For each function $f : A \to B$ below, prove/disprove if f is surjective and/or injective.

(a) (6 points) $f(x) = 2^x$, $A = \mathbb{Z}$, $B = \mathbb{R}^+$.

(b) (6 points) $f(x, y) = x - y$, $A = \mathbb{Z} \times \mathbb{Z}^+$, $B = \mathbb{Z}$.

(c) (6 points) $f(x) = \begin{cases} 2 - x & \text{if } x \leq 1 \\ 1/x & \text{otherwise} \end{cases}$, $A = B = \mathbb{R}$.

Problem 4 (20+2 points)
Let f and g be functions from a finite set X to itself.

(a) (10+1 points) Prove or disprove: if $f \circ g$ is bijective, then f and g are both bijective.

(b) (10+1 points) Prove or disprove: if X is infinite, then the statement in part (a) is true.

Problem 5 (15+1 points)
Suppose 170 students are standing in a line, all facing to the right, and each student has a unique birthday. A subsequence is a subset of students selected from the line; a subsequence is *increasing* (*decreasing*) if each student in the subsequence is older (younger) than the previous student in the subsequence. Prove that there exists a subsequence of length 14 that is either increasing or decreasing.

(Recall that a subsequence obeys the original order, but is not necessarily contiguous: $(1, 4, 5)$ is a subsequence of $(1, 6, 4, 5, 3)$, but $(6, 1)$ is not.)

Problem 6 (20+2 points)
Let R be a relation on $\mathbb{Z}^+ \times \mathbb{Z}^+$ defined as follows: $(a, b)R(c, d)$ if and only if $ad = bc$.

(a) (10+1 points) Prove that R is an equivalence relation.

(b) (10+1 points) Define a bijection from the set of equivalences classes induced by R to the set of positive rationals, and prove that this function is indeed bijective.