COMPSCI 230: Discrete Mathematics for Computer Science March 4, 2019

Lecture 14

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1 Overview

In Lecture 13, we introduced the notion of perfect matchings in bipartite graphs. We also saw Hall's Theorem, which tells us when a bipartite graph has a perfect matching. In this lecture, we give a proof of Hall's Theorem.

2 Hall's Theorem

In this section, we re-state and prove Hall's theorem. Recall that in a bipartite graph $G = (A \cup B, E)$, an A-perfect matching is a subset of E that matches every vertex of A to exactly one vertex of B, and doesn't match any vertex of B more than once.

Theorem 1 (Hall 1935)**.** *A bipartite graph $G = (A \cup B, E)$ has an A-perfect matching if and only if the following condition holds:*

$$\forall S \subseteq A. \, |N(S)| \geq |S|,$$

where $N(S) = \{v \in B : \exists u \in S. \{u, v\} \in E.\}$.

Remark 1: If $|A| = |B|$, then a matching is A-perfect if and only if it is perfect. So, in this case, Hall's theorem tells us when a *perfect* matching exists. On the other hand, if $|A| \neq |B|$, then G cannot contain a perfect matching because every edge of a matching pairs one vertex of A with exactly one vertex of B. However, if $|A| < |B|$, then G may still contain an A-perfect matching.

Remark 2: Theorem 1 is of the form $P \leftrightarrow Q$, where P is the proposition "G has an A-perfect matching" and Q is known as *Hall's condition*. In general, $P \to Q$ states that Q is a *necessary* condition for P. (To see why, consider the contrapositive $\neg Q \to \neg P$.) Furthermore, $Q \to P$ states that Q is a *sufficient* condition for P. Thus, Hall's theorem states that Hall's condition is a necessary and sufficient condition for a bipartite graph to have an A-perfect matching.

Proof. We now begin the proof of Theorem 1.

Hall's condition is necessary: Assume that G has an A-perfect matching, which we denote by M. Let S be an arbitrary subset of A. Since M is an A-perfect matching, M matches every vertex of S to exactly one vertex of B, and no vertex of B is matched more than once. So if we restrict G to the edges in M, the vertices of S each have a distinct neighbor in $N(S)$. Since $N(S)$ is defined using *all* the edges of G and M is only a *subset* of E, this implies $|N(S)| \geq |S|$.

Hall's condition is sufficient: We will construct an A-perfect matching M by proceeding with induction on $|A|$, assuming G satisfies Hall's condition.

Base case: $|A| = 1$. Let a denote the sole vertex of A. Hall's condition tells us $|N(\{a\})| \geq 1$, which means a has at least one neighbor. We can set $M = \{\{a, b\}\}$ where b is any neighbor of a.

Then, M is an A-perfect matching: every vertex of A is matched, and no vertex of B is matched more than once.

Inductive hypothesis (IH): Assume that for all k such that $1 \le k \le |A| - 1$, any bipartite graph $H = (C \cup D, F)$ satisfying $|C| = k$ has a C-perfect matching if and only if H satisfies Hall's condition on C, i.e., $\forall S \subseteq C.\, |N(S)| \ge |S|$.

Inductive step: We will now construct an A-perfect matching M in G, starting with $M = \emptyset$. Note that when G satisfies Hall's condition, there are two possible cases: the inequality is strict for every S that is a strict subset of A (i.e., $\forall S \subset A.\, |N(S)| > |S|$), or there exists at least one $S \subset A$ such that $|N(S)| = |S|$.

(i) In the first case, since $|N(S)|$ and $|S|$ are integers, we can assume

$$\forall S \subset A.\, |N(S)| \ge |S| + 1. \tag{1}$$

We claim that the following procedure returns a perfect matching:

1. Let u be an arbitrary vertex of A, and add *any* edge $e = \{u, v\}$ of E to M.
2. Remove u, v, and all edges incident to u or v from G to construct graph G'.
3. By the IH, G' has a matching M' that is $(A \setminus \{u\})$-perfect.
4. Add the edges of M' to M, and return M.

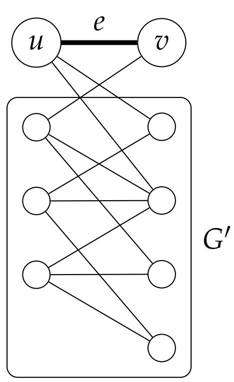

Figure 1: The original graph G with an arbitrary edge $\{u, v\}$ added to M (bold). The graph G' comprises the remaining vertices.

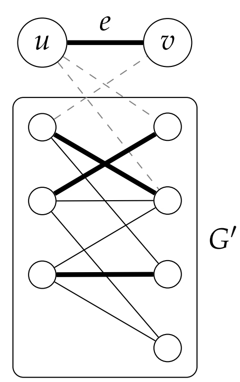

Figure 2: Applying the IH on G' yields a matching M' that is $(A \setminus \{u\})$-perfect. The edges of the final matching M are in bold.

For this procedure to be correct, we have to show several properties. First, to show that Step 1 is valid, we need to establish that the degree of any vertex $u \in A$ is at least 1. If this does not hold, then $N(\{u\}) = 0$, while $|\{u\}| = 1$, thereby violating Hall's condition for $S = \{u\}$.

Next, we show that G' satisfies Hall's condition so that Step 3 is valid. Let S' be any subset of $A \setminus \{u\}$, and let $N'(S')$ denote its neighbors in G' (so $N'(S') \subseteq B \setminus \{v\}$). Notice that $|N(S')| - 1 \ge |S'|$ because of (1). Since we only removed one vertex of B to construct G', $|N'(S')| \ge |N(S')| - 1$. Taken together, these inequalities imply $|N'(S')| \ge |S'|$, as desired.

Now we must show that $M = M' \cup \{\{u, v\}\}$ is an A-perfect matching. Since M' matches every vertex of $A \setminus \{u\}$ and e matches u, every vertex of A is indeed matched by M. Furthermore, vertex v is matched once because G' excludes v, and the vertices of $B \setminus \{v\}$ are matched at most once because M' is a matching. Thus, M is an A-perfect matching.

(ii) In the second case, we assume there exists $S \subset A$ such that $|N(S)| = |S|$. We claim that the following procedure returns a perfect matching:

1. Partition A into S and $\overline{S} = A \setminus S$ and B into $N(S)$ and $\overline{N(S)} = B \setminus N(S)$.
2. Let $G_1 = (S \cup N(S), E_1)$ where E_1 denotes the edges of G among $S \cup N(S)$. By the IH, G_1 has a matching M_1 that is S-perfect.
3. Let $G_2 = (\overline{S} \cup \overline{N(S)}, E_2)$, where E_2 denotes the edges of G among $\overline{S} \cup \overline{N(S)}$. By the IH, G_2 has a matching M_2 that is $\overline{S}$-perfect.
4. Let $M = M_1 \cup M_2$, and return M.

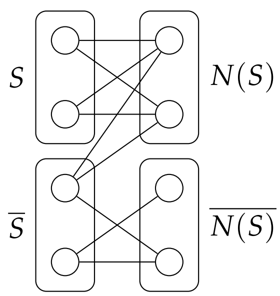

Figure 3: Partitioning A and B according to S and $N(S)$; notice that $|N(S)| = |S|$.

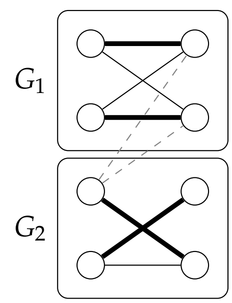

Figure 4: Applying the IH on G_1 and G_2 and returning the union of the two matchings.

As in the previous case, we must prove that G_1 and G_2 satisfy Hall's condition so that Step 2 and Step 3 are valid. The graph G_1 is easier to handle: observe that there are no edges from S to $\overline{N(S)}$. Thus, for any subset T of S, its neighborhood in G_1 is exactly the same as its neighborhood in G. Since G satisfies Hall's condition, this implies G_1 also satisfies Hall's condition.

Showing that G_2 also satisfies Hall's condition is slightly trickier. For contradiction, suppose G_2 violates Hall's condition. This means there exists $X \subseteq \overline{S}$ such that $|Y| < |X|$, where Y denotes the neighborhood of X in G_2, i.e., $Y = N(X) \cap \overline{N(S)}$. Now consider the set $S \cup X$. This is a subset of A, so since G satisfies Hall's condition, we know that

$$|N(S \cup X)| \geq |S \cup X|.$$

Furthermore, since S and X are disjoint, $|S \cup X| = |S| + |X|$. Also, notice that the only neighbors of $S \cup X$ contained in $\overline{N(S)}$ are the neighbors of X, i.e., $N(S \cup X) = N(S) \cup Y$. Finally, since $N(S)$ and Y are disjoint, we know that $|N(S) \cup Y| = |N(S)| + |Y|$. Putting this all together, we get

$$\begin{aligned}
|N(S \cup X)| &= |N(S) \cup Y| && \text{(definitions of } S, X, Y\text{)} \\
&= |N(S)| + |Y| && (N(S) \text{ and } Y \text{ are disjoint}) \\
&= |S| + |Y| && \text{(defining property of } S\text{)} \\
&< |S| + |X| && \text{(defining property of } X\text{)} \\
&= |S \cup X|. && (S \text{ and } X \text{ are disjoint})
\end{aligned}$$

Thus, the set $S \cup X$ violates Hall's condition in the original graph G because $|N(S \cup X)| < |S \cup X|$. This concludes the proof that G_2 satisfies Hall's condition.

Now that we know G_1 and G_2 satisfy Hall's condition, we must show that $M = M_1 \cup M_2$ is an A-perfect matching. Since M_1 is S-perfect and M_2 is $\overline{S}$-perfect, we know that M matches every vertex of $S \cup \overline{S} = A$. Furthermore, no edges of M_1 and M_2 share an endpoint because M_1 and M_2 were obtained from two disjoint graphs G_1 and G_2. Thus, no vertex of B is matched twice by M, so M is an A-perfect matching. $\square$

3 Summary

In this lecture, we proved Hall's theorem, one of the most well-known results in discrete mathematics. The proof uses induction in a manner that is more complicated than typical induction proofs we have seen.