COMPSCI 230: Discrete Mathematics for Computer Science	March 6, 2019
Lecture 15	
Lecturer: Debmalya Panigrahi	*Scribe: Kevin Sun*

1 Overview

In this lecture, we introduce the basic definitions related to graph coloring and connectivity. These ideas are fundamental in graph theory, which is a core topic of discrete mathematics.

2 Graph Coloring

2.1 Scheduling Final Exams

Every semester, the registrar must schedule final exams in a way so that no student must take two exams at the same time. We can begin modeling this situation by constructing a graph $G = (V, E)$. In this case, the vertex set V corresponds to the set of all classes (one vertex per class). The set E is defined as follows: there is an edge between u and v if there exists at least one student taking both class u and class v. The goal is to assign each vertex to a time slot so that the two endpoints of every edge are assigned to different slots.

The simplest solution is to assign each final exam to a unique time slot. If there are n classes, this results in n time slots, but this may lead to an extremely long final exam period. So we want a solution that is not only feasible, but also minimizes the number of total time slots.

One way to visualize the notion of "assigning a time slot" is to consider each time slot as a color. In other words, given G, we must color every vertex of G so that the two endpoints of every edge receive different colors. Fig. 1 gives an example of coloring a graph with 5 vertices using 3 colors, and Fig. 2 colors the same graph using only two colors.

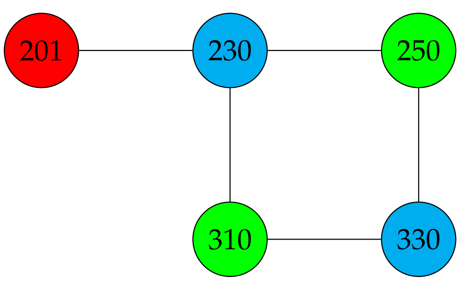

Figure 1: A graph G with 5 vertices corresponding to 5 classes. Each of the three colors represents a distinct time slot.

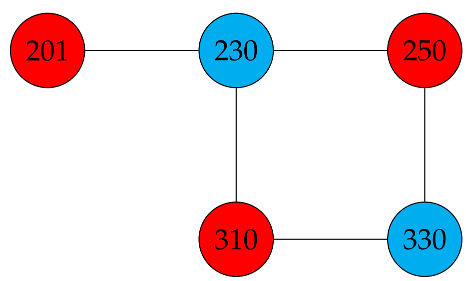

Figure 2: The same graph from Fig. 1, colored using only two colors.

Definition 1. *Let k be a positive integer. A graph is* k-colorable *if it is possible to assign each vertex to one of k colors such that the two endpoints of every edge are assigned different colors.*

When a coloring properly assigns the endpoints of an edge to different colors, we often say that the coloring *respects* the edge. If a coloring respects every edge of the graph, then the coloring is *proper* or *valid*.

The graph shown in Fig. 2 is 2-colorable, since every edge has a red endpoint and a blue endpoint. Notice that Fig. 1 shows that the same graph is 3-colorable—in general, if a graph is k-colorable, then it is also ℓ-colorable for any $\ell \geq k$. We will now prove a simple observation regarding graphs that are 2-colorable.

Observation 1. *Let G be a graph. Then G is 2-colorable if and only if G is bipartite.*

Proof. Let G be a 2-colorable graph, which means we can color every vertex either red or blue, and no edge will have both endpoints colored the same color. Let A denote the subset of vertices colored red, and let B denote the subset of vertices colored blue. Since all vertices of A are red, there are no edges within A, and similarly for B. This implies that every edge has one endpoint in A and the other in B, which means G is bipartite.

Conversely, suppose G is bipartite, that is, we can partition the vertices into two subsets V_1, V_2 every edge has one endpoint in V_1 and the other in V_2. Then coloring every vertex of V_1 red and every vertex of V_2 blue yields a valid coloring, so G is 2-colorable. □

Thus, Observation 1 tells us that the graph in Fig. 2 is bipartite. Indeed, by observing Fig. 3, it becomes even clearer that this graph is bipartite.

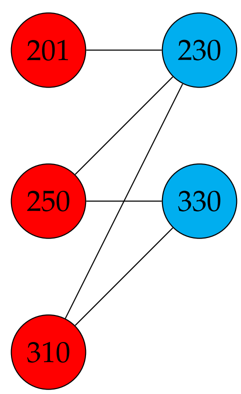

Figure 3: The same graph and coloring from Fig. 2, with the vertices both colored and rearranged to further illustrate that it's bipartite.

Color cycle graphs: We now consider graph colorings restricted to the set of cycle graphs. Recall that C_n denotes the cycle graph with n vertices when $n \geq 3$; this is the graph with vertex set is $V = \{1, 2, \ldots, n\}$ and edge set $E = \{\{i, i+1\} : i = 1, 2, \ldots, n-1\} \cup \{1, n\}$.

From Figs. 4, 5, and 6, we can see that C_4 is 2-colorable, but C_3 and C_5 are not. The leads us to our second observation, which characterizes when a cycle is 2-colorable.

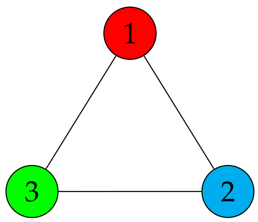

Figure 4: The graph C_3 with a valid 3-coloring. Notice that C_3 is not 2-colorable.

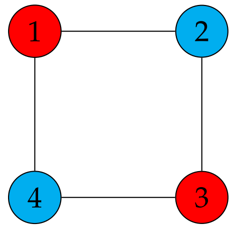

Figure 5: The graph C_4 with a valid 2-coloring.

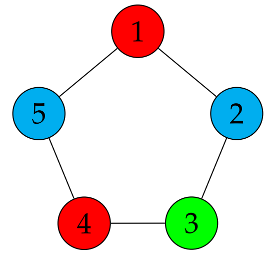

Figure 6: The graph C_5 with a valid 3-coloring. Notice that C_5, like C_3, is also not 2-colorable.

Observation 2. *The cycle graph C_n on n vertices is 2-colorable if and only if n is even.*

Proof. Let $V = \{1, 2, \ldots, n\}$ denote the vertex set. If n is even, then we can color vertex k red if k is odd, and blue if k is even. Notice that every edge has an odd endpoint and an even endpoint, which means that this coloring is valid.

Now suppose n is odd, so $n = 2k + 1$. For contradiction, suppose the vertices are colored red and blue such that every edge has a red endpoint and a blue endpoint. Without loss of generality, suppose vertex 1 is colored red. This implies vertex 2 must be colored blue, which implies vertex 3 is colored red, and so on. Thus, the vertices $\{1, 3, \ldots, 2k-1, 2k+1\}$ are colored red, but the edge $\{1, 2k+1\}$ exists in the graph and both endpoints are colored red. This contradicts the assumption that the coloring is valid. $\square$

Coloring complete graphs: Now we restrict our attention to coloring complete graphs. Recall that the complete graph on n vertices, denoted K_n, is contains an edge between every pair of (distinct) vertices. (Note that $K_3 = C_3$.) As we noted earlier, any graph on n vertices has a valid coloring with n colors, i.e., is n-colorable. For a complete graph, this is indeed the *only* valid coloring in the sense that there is no valid coloring that uses less than n colors.

Observation 3. *The complete graph K_n on n vertices is not $(n-1)$-colorable.*

Proof. Consider any color assignment on the vertices of K_n that uses at most $n-1$ colors. Since there are n vertices, there exist two vertices u, v that share a color. However, since K_n is complete, $\{u, v\}$ is an edge of the graph. This edge has two endpoints with the same color, so this coloring is not valid, so K_n is not $(n-1)$-colorable. $\square$

Coloring Planar Graphs: Finally, we restrict our attention to coloring planar graphs (defined later). Consider a map of the countries of Europe, and suppose we want to color the countries such that any two countries that share a border are colored different colors. For example, France and Germany should be assigned different colors, but France and Sweden can be colored the same color.

We can represent this problem as a graph coloring problem: the graph contains a vertex for every country, and an edge exists between two vertices if their corresponding countries share a border. Thus, coloring the regions in the map corresponds to coloring the vertices of the graph, and no two adjacent regions (vertices) should be colored the same color.

A graph is *planar* if it can be drawn on a plane (e.g., on a sheet of paper) such that no two edges cross each other. For example, the graph in Fig. 3 is planar, and the drawing in Fig. 2 illustrates this. It's intuitively clear that every map can be represented as a planar graph. The theorem below states that when coloring a map, or any planar graph, we only need 4 colors.

Theorem 4 (Four Color Theorem). *If G is a planar graph, then G is 4-colorable.*

The proof of Theorem 4 is beyond the scope of this course, so we omit it. We will note, however, that the theorem was originally conjectured in 1852. Multiple proofs were proposed, but all were shown to be flawed until 1976, when Appel and Haken announced a computer-assisted proof. Their proof involved a large number of cases that are too tedious to check by hand. Since their proof was published, the last 40 years has seen several more "verifiable" proofs of this theorem, where the number of graphs that need to be checked manually has significantly reduced over time.

3 Connectivity and Cycles

Now we will study a different property of graphs known as connectivity, and we will begin with some basic definitions. Throughout this section, we let $G = (V, E)$ be a graph.

3.1 Walks, paths, and connected components

We now formalize the notion of traversing edges in a graph. Intuitively, we can think of the following definitions as capturing a time-dependent process of moving along the edges of a graph.

Definition 2. *A* walk *in G is a sequence of vertices such that there exists an edge between every two consecutive vertices in the sequence.*

Definition 3. *A* path *in G is a walk that does not repeat any vertices.*

Remark: If a walk does not repeat vertices, then it also does not repeat edges. To see this, suppose the edge $\{u, v\}$ is traversed more than once in a walk. Then u and v each appear at least twice, so the walk also repeats vertices.

Furthermore, we can show the following: if there is a walk from s to t, then there is a path from s to t. Let w be a walk from s to t; we will eliminate repeated vertices in w to create a path. If u is a vertex that appears multiple times in w, then consider removing all vertices between the first and last appearance of u. The resulting sequence is still a walk, and this process can be repeated until there are no repeated vertices.

With these definitions, we can formalize the definition of connectivity in graphs. Intuitively, a graph is connected if it contains one "piece," but as we shall see, this can be captured through the use of an appropriately defined equivalence relation on V.

Definition 4. *Two vertices u and v are* connected *if there exists a path that starts at u and ends at v.*

Consider the relation R on V defined as follows: for all $u, v \in V$, the pair (u, v) is in R if and only if u and v are connected. It is straightforward to verify that R is an equivalence relation:

- Reflexive: every vertex u is connected to itself—simply take walk that starts at u and contains no edges. Thus, $(u, u) \in R$ for every $u \in V$.

- Symmetric: if there's a path from u to v, then that same sequence of vertices in reverse gives a path from v to u. Thus, if $(u, v) \in R$, then $(v, u) \in R$.
- Transitive: if there's a path p_1 from u to v and a path p_2 from v to w, then we can join these two paths at v to create a walk from u to w. As we discussed earlier, this walk can be used to obtain a path from u to w. This implies R is transitive.

Now recall that every equivalence relation induces a partition on the underlying set, and the blocks of the partition are the equivalence classes of the relation. Let us consider the partition that R induces on V: let u be any vertex, and let $[u] = \{v : (u, v) \in R\}$ be the equivalence class of u. Then $[u]$ contains all the vertices reachable from u; this is known as the *connected component* (or simply *component*) of u. In general, each equivalence class is known as a connected component (or component), and together, they are known as the connected components (or components) of G.

Definition 5. *A graph is* connected *if every pair of vertices is connected.*

Thus, an equivalent definition of G being connected is that G contains exactly one connected component. In this case, the partition induced by R only has one block, which is equal to V.

3.2 Bipartite graphs and odd cycles

In this section, we will see another important and well-known characterization of bipartite graphs.

Definition 6. *A* cycle *in G is a walk with at least 3 vertices such that all vertices are distinct, except the first vertex is equal to the last.*

As we might expect, the notion of a cycle appearing in a graph is closely related to the set of cycle graphs. An equivalent perspective is the following: G contains a cycle of length k if there exist $u_1, u_2, \ldots, u_k \in V$ such that $\{u_i, u_{i+1}\} \in E$ for every $i \in \{1, 2, \ldots, k-1\}$, and $\{u_1, u_k\} \in E$ as well. We say that a cycle is *odd* if the number of vertices (equivalently, number of edges) is odd; else, the cycle is said to be even.

Theorem 5. *A graph is bipartite if and only if it does not contain an odd cycle.*

Proof. Suppose G has an odd cycle $C = (u_1, u_2, \ldots, u_{2k}, u_{2k+1}, u_1)$ where k is a positive integer. This cycle is simply the graph C_{2k+1}, which from Observation 2 is not 2-colorable. This implies that G is not 2-colorable either, because this subset of V cannot be colored. Thus, by Observation 1, G is not bipartite.

Conversely, suppose G does not contain an odd cycle. We will prove that G is 2-colorable by induction on $|E|$. From Observation 1, this implies that G is bipartite.

Base case: If $|E| = 0$, then we can color the vertices arbitrarily, so the graph is 2-colorable.

Inductive hypothesis: For any graph G' with fewer than $|E|$ edges, if G' contains no odd cycle, then G' is 2-colorable.

Inductive step: Our plan follows the usual proof template of induction on graphs. Given a graph G with $|E|$ edges, we will reduce the number of edges to create a graph G', apply the induction hypothesis on G' to obtain a 2-coloring of G', and use this to construct a 2-coloring of G.

1. Remove any edge $e = \{u, v\}$ from G to create G', that is, $G' = (V, E \setminus \{e\})$.

2. Since G contains no odd cycle, neither does G' (because *removing* an edge cannot *create* a cycle). Thus, we can apply the IH on G' to obtain a 2-coloring of G'. Suppose this coloring assigns each $u \in V$ to a color $c(u) \in \{R, B\}$ (red or blue).

3. We must now use the coloring function $c : V \to \{R, B\}$, which is valid for G', to construct a coloring for the original graph G. Recall that G' is simply G with some edge $\{u, v\}$ removed; we proceed by two possible cases: u and v are either connected in G', or they are not.

   i. If u and v are connected in G', then there's a path $p = (u, u_1, u_2, \ldots, u_k, v)$, where $k \in \mathbb{Z}^+$, from u to v in G'. Notice that p is contained in G, so p together with the edge $\{u, v\}$ forms a cycle in G. Since G contains no odd cycle, the path p must have an even number of vertices, which means k is even. Thus, the vertices $u, u_2, \ldots, u_k$ are all colored red (say), and the vertices $u_1, u_3, \ldots, v$ are colored blue. This means $c(u) \neq c(v)$, so the coloring c also respects the edge $\{u, v\}$ (in addition to the remaining edges of G). See Fig. 7 for an illustration.

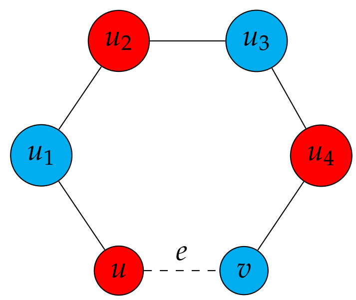

Figure 7: The case where u and v are connected in G' (solid lines). Since this path along with the missing edge $e = \{u, v\}$ (dashed) forms a cycle, and G has no odd cycle, the path must have an even number of vertices.

   ii. If u, v are not connected in G', then u and v are in two different connected components of G'. If it is the case that $c(u) \neq c(v)$, then c is still a valid coloring for G, so we are done. However, it might be possible that $c(u) = c(v)$. In this case, let F denote the vertices in the same connected component as v. We shall construct a new coloring $d : V \to \{R, B\}$ that "flips" the colors in F and keeps the others untouched. More formally, d is defined as follows: if $u \in F$, then set $d(u)$ as the opposite of $c(u)$; otherwise, set $d(u) = c(u)$.

   Notice that d is still valid for every edge x of G': if the endpoints of x are not in F, then d follows c, which respects x; otherwise, d swaps the colors assigned by c, so d still assigns different colors to the endpoints of x. Furthermore, $d(u) \neq d(v)$, so d also respects the edge $\{u, v\}$. Thus, d is a valid 2-coloring for G. See Figs. 8 and 9 for an illustration.

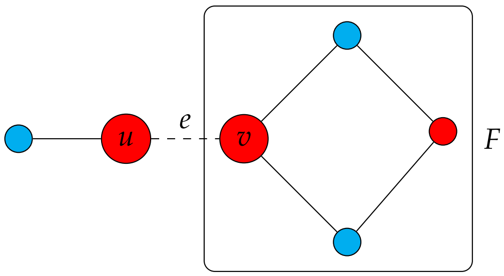

Figure 8: The case where $c(u) = c(v)$. We will modify this coloring by flipping all colors of F while leaving the other colors unchanged.

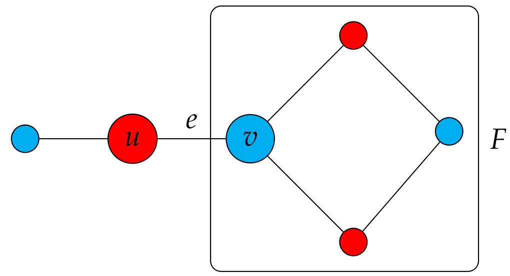

Figure 9: The modified coloring d obtained from Fig. 8. Notice that d, like c, is valid for F, and but d also satsifies $d(u) \neq d(v)$, so d is also valid for $e = \{u, v\}$.

□

The following statement summarizes Observation 1 and Theorem 5:

$$G \text{ is bipartite } \leftrightarrow G \text{ is 2-colorable } \leftrightarrow G \text{ does not contain an odd cycle.}$$

4 Summary

In this section, we saw the fundamental concepts and definitions related to coloring and connectivity in graphs. We proved multiple observations, as well as a more substantial theorem that describes the relationship between bipartite graphs and odd cycles.