

Lecture 24

Lecturer: Debmalya Panigrahi

Scribe: Erin Taylor

1 Overview

In this lecture, we continue our study of probability. We focus on conditional probabilities and independence.

2 Conditional Probability

Recall our definition of conditional probability, when we want to know the probability of the event A given B :

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

We are asking for the probability of event A after restricting our sample space to event B , see the following diagram in Figure 1. Event $A \cap B$ has a small probability, but once we restrict to only event B instead of the entire sample space, it may be a larger fraction and therefore a higher probability.

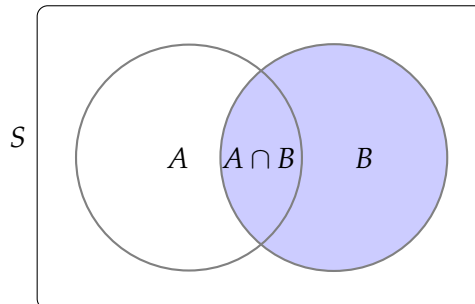


Figure 1: Sample space S , shaded blue area showing restricting space to event B .

Example 1: Let's revisit the Monty Hall problem now that we know the concept of conditional probability. Recall that there are 3 closed doors A , B , and C for the guest to pick and behind one door is a car, while behind the other two are goats. After the guest picks a door, the host opens one of the two remaining doors to reveal a goat. Then, the guest is given the option to switch the door they picked. We saw that the guest is more likely to win by switching. Let's define some events formally.

X : guest wins car by switching

Y : car is at location A and there is a goat at B

Z : guest picks door A and host reveals a goat at B

Event Y happens when the car is at location A , since both other locations will necessarily have goats. Thus, $\Pr(Y) = 1/3$. Event X and Y both happen when the guest does not originally pick door A , but then switches to door A . Thus, the guest could pick door B or C . This could happen 2 ways with probability $\Pr(X \cap Y) = 2/9$. Now we can compute the probability of the guest winning by switching given that the car is at A :

$$\Pr(X|Y) = \frac{\Pr(X \cap Y)}{\Pr(Y)} = \frac{2/9}{1/3} = 2/3.$$

Let's consider event Z . There are two ways for this event to occur. The location of the car is A and the guest picks A , then the host reveals goat at B is one way with probability $1/18$. The location of the car is at A , the guest picks C , then host reveals goat at B occurs with probability $1/9$. Thus,

$$\Pr(X|Z) = \frac{P(X \cap Z)}{P(Z)} = \frac{1/9}{1/9 + 1/18} = 2/3.$$

The calculation of $\Pr(X|Y)$ includes the outcome that the host opens door C , which included some extraneous outcome in our calculation. In reality, we are interested in the probability of winning by switching given the guest picks door A and the host opens door B . This aligns with what we calculated last time— that the guest's best strategy is to switch.

Let's consider another example of conditional probability leading to a counter-intuitive result. **Example 2:** Suppose we have a test that tells us whether a person is sick with a high degree of accuracy. For a person who is healthy, the test will likely be negative and for a person who is sick, the test will likely be positive. We define the following events.

$$\begin{aligned} A : \text{test is positive} & \quad \bar{A} : \text{test is negative} \\ B : \text{person is healthy} & \quad \bar{B} : \text{person is sick} \end{aligned}$$

We have the following tree in Figure 2 showing the probability that someone is healthy and the probability of each test result. A positive test result for a healthy person is known as a false positive, and a negative test for a sick person is known as a false negative.

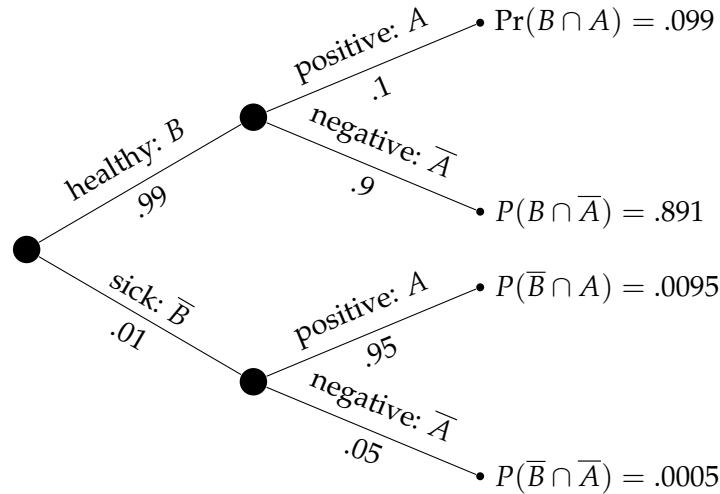


Figure 2: Tree Diagram for Example 2.

Now, if we run the test and the test is positive, intuitively we expect that the person is sick. In other words, we expect the number of false positives to be low. That is, we expect $\Pr(B|A)$ to be low and $\Pr(\bar{B}|A)$ to be high. Let's see if this is the case.

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{.099}{.1085} = .91$$

$$\Pr(\bar{B}|A) = \frac{\Pr(A \cap \bar{B})}{\Pr(A)} = \frac{.0095}{.1085} = .09$$

What happened? If we were to test a random person and the test is positive, it is more likely that they are healthy than they are sick. This is the result of the fact that the vast majority of people are healthy. Fortunately, medical tests are not usually run on random people, but on those showing symptoms of being sick!

When considering two events, we may not always know the probability of their intersection. We will introduce two rules that are useful for calculating conditional probabilities in such cases.

1. **Bayes' Rule:** Suppose we have two events A and B . Bayes' Rule relates the conditional probabilities $P(A|B)$ and $P(B|A)$.

$$\Pr(A|B) = \frac{\Pr(B|A) \Pr(A)}{P(B)}$$

Note that $\Pr(A \cap B) = \Pr(B|A) \Pr(A) = \Pr(A|B) \Pr(B)$ by our definition of conditional probability, so this rule is straightforward to derive from there.

2. **Law of Total Probability:**

$$\Pr(A) = \Pr(A|B) \Pr(B) + \Pr(A|\bar{B}) \Pr(\bar{B})$$

This law is also intuitive, the total probability of event A is the sum of the probability of $A \cap B$ and $A \cap \bar{B}$.

We can also combined the two rules:

$$\Pr(A|B) = \frac{\Pr(B|A) \Pr(A)}{\Pr(B|A) \Pr(A) + \Pr(B|\bar{A}) \Pr(\bar{A})}.$$

Example 3: Suppose we want to know the probability of Duke winning the NCAA tournament. We define the following events.

A : Duke wins the NCAA tournament

B : Duke beats UNC in the ACC tournament

Suppose we are given that $\Pr(B) = .75$, $\Pr(A|B) = .99$, and $\Pr(A|\bar{B}) = .25$. Now, if we know that Duke won the NCAA tournament, what is the probability they beat UNC? We can use the above formula:

$$\Pr(B|A) = \frac{\Pr(A|B) \Pr(B)}{\Pr(A|B) \Pr(B) + \Pr(A|\bar{B}) \Pr(\bar{B})} = \frac{(.99)(.75)}{(.99)(.75) + (.25)(.25)} \approx 0.922$$

2.1 Simpson's Paradox

Suppose we have two schools within a university: School A and School B. The school admits 100 students per year, and last year they admitted 51 men and 49 women. Is this school admissions process biased against women? Let's consider the admissions breakdown by school.

| School | Men | Women |
|---------|--------|--------|
| A | 1/10 | 40/90 |
| B | 50/90 | 9/10 |
| Overall | 51/100 | 49/100 |

It seems that each individual school is biased against men, but overall the university is biased against women. How could this be possible? This is a result of the fact that one school is more selective than the other. More women applied to the more selective school, so overall fewer women gain admission to the university. In the language of probabilities, the conditional probability of acceptance of men in both schools is smaller than that of women, but the unconditional probability of acceptance of men is higher than that of women.

3 Independence

Recall that two events A and B are independent if knowing the outcome of B does not affect the probability of A , and vice versa. Formally, A and B are independent of B if $\Pr(A|B) = \Pr(A)$. Equivalently, $\Pr(A \cap B) = \Pr(A) \Pr(B)$. We will consider the case when we have more than 2 events.

Definition 1. Suppose we have events A_1, \dots, A_n and k is a positive integer such that $k \leq n$. These events are said to be k -wise independent if, for any subset with size $\leq k$, the subsets are independent.

$$\Pr(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_j}) = \Pr(A_{i_1}) \Pr(A_{i_2}) \dots \Pr(A_{i_j}) \text{ for any } j \leq k.$$

Definition 2. Suppose we have events A_1, \dots, A_n . These events are said to be mutually independent if they are n -wise independence.

Intuitively, a set of events is mutually independent if the probability of each event is the same no matter which of the other events has occurred.

Example 4: Suppose we have three fair coins, and we toss all of them such that each toss is independent. We will define the following events:

A_1 : Coins 1 and 2 have the same result

A_2 : Coins 2 and 3 have the same result

A_3 : Coins 1 and 3 have the same result

We will show that these events are 2-wise independent, often called *pairwise independent*. $\Pr(A_1) = \Pr(A_2) = \Pr(A_3) = 1/2$. Then,

$$\Pr(A_1 \cap A_2) = 1/4$$

However, consider the event $A_1 \cap A_2 \cap A_3$. There are two ways for this to happen: all heads or all tails. So this occurs with probability $1/4$. However, $\Pr(A_1) \Pr(A_2) \Pr(A_3) = 1/8$. Thus, these events are not 3-wise independent. Note that $\Pr(A_3 | A_1 \cap A_2) = 1$.

4 Summary

In this lecture, we revisited the Monty Hall problem using the perspective of conditional probability. We also introduced Bayes' rule and the Law of total probability. We defined independence of events, and gave some examples where our intuition does not match up with the calculations of probabilities.