

## Lecture 6

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## 1 Overview

This lecture begins our third module: Sets, Maps, and Sequences. (The first module was Proofs, and the second was Mathematical Logic.) In this lecture, we give an overview of sets, operators on sets, rules that we can apply, and the set builder notation.

## 2 Sets

As we saw in Lecture 1 with Russell's paradox, giving a precise definition of a *set* is a notoriously difficult problem. However, for the purposes of this class, we can simply think of a set as a collection of objects known as "elements" of the set. Note that a set never has repeated elements, and elements of a set are not ordered. For example,

$$S = \{1, 4, 3\}$$

is a set containing 3 elements. Equivalent versions of  $S$  include  $\{1, 1, 4, 3\}$  and  $\{4, 3, 1\}$ .

**Definition 1.** If  $A$  and  $B$  are sets, then  $A$  is a subset of  $B$  (denoted by  $A \subseteq B$ ) if the following proposition is true:

$$x \in A \Rightarrow x \in B.$$

This definition is fairly intuitive: a subset of a set is a new set whose elements all belong to the original set. For example,  $\{1\}$  is a subset of  $S$  (defined above), but  $\{1, 2\}$  is not. Also, note that " $1$ " is not a subset of  $S$ , because " $1$ " on its own is not a set; sets are defined with curly brackets.

If  $A$  is the empty set (which we denote by  $A = \emptyset$ ), that is,  $A$  contains no elements, then  $A$  is a subset of any set. This is because the implication above that defines "subset" is vacuously true:  $x \in A$  is false for any  $x$ , and  $\text{FALSE} \Rightarrow Q$  is TRUE regardless of the value of  $Q$ .

**Definition 2.** If  $A$  and  $B$  are sets, then  $A$  is a proper subset of  $B$  (denoted  $A \subset B$ ) if  $A$  is a subset of  $B$  and  $A \neq B$ .

In other words,  $A$  is a proper subset of  $B$  if every element of  $A$  is in  $B$ , and  $B$  contains at least one element that is not in  $A$ . Note that the empty set is a proper subset of any non-empty set, and in the above example,  $\{1\}$  is a proper subset of  $S$ .

**Definition 3.** The power set of a set  $S$  is a set of all the subsets of  $S$ , often denoted by  $2^S$ .

As discussed above, the empty set is a subset of any set. Furthermore, any set is a subset of itself. Thus, the power set of  $S$  is the following:

$$2^S = \{\emptyset, \{1\}, \{4\}, \{3\}, \{1, 4\}, \{1, 3\}, \{4, 3\}, S\}.$$

The set  $2^S$  has 8 elements, each of which is a subset of  $S$ .

**Definition 4.** The cardinality of a finite set is the number of elements in it, often denoted by  $|S|$ .

(A finite set is simply a set that contains a finite number of elements.) Continuing with our example, we have  $|S| = 3$  and  $|2^S| = 8$ . It is unclear how we can define cardinality of an infinite set, but as we will see how we can still impose a notion of size on infinite sets in a later lecture.

## 2.1 Operators on Sets

In propositional logic, we had operators such as  $\wedge$  and  $\vee$  that acted on propositional variables. Similarly, we will use three basic operators on sets, and the definition of each agrees with our intuitive understanding of the corresponding words. We shall keep the following example in mind as a running example, where the universe is the set of positive integers less than 7:

$$A = \{1, 3, 4\}, \quad B = \{2, 6\}.$$

1. Union: The *union* of two sets  $A$  and  $B$ , denoted  $A \cup B$ , is a set defined as follows:

$$x \in A \cup B \Leftrightarrow (x \in A \vee x \in B).$$

In our example,  $A \cup B = \{1, 2, 3, 4, 6\}$ .

2. Intersection: The *intersection* of two sets  $A$  and  $B$ , denoted  $A \cap B$ , is a set defined as follows:

$$x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B).$$

In our example,  $A \cap B = \emptyset$ .

3. Complement: The *complement* of a set  $A$ , denoted by  $\overline{A}$  or  $A^c$ , is defined as follows:

$$x \in \overline{A} \Leftrightarrow x \notin A.$$

(The notation  $x \notin A$  is shorthand for  $\neg(x \in A)$ ; in other words, it means  $x$  is not an element of  $A$ .) In our example,  $\overline{A} = \{2, 5, 6\}$  and  $\overline{B} = \{1, 3, 4, 5\}$ .

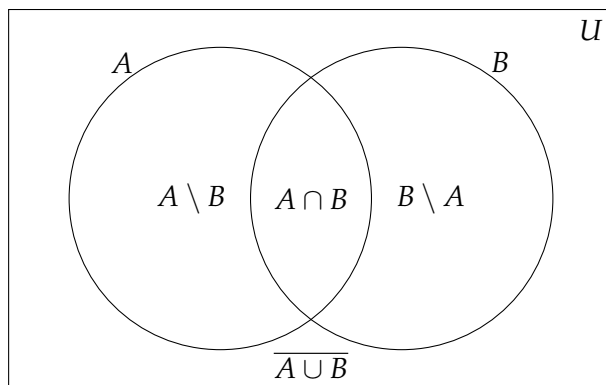
Now we introduce a fourth operator: the *difference* of sets  $A$  and  $B$ , denoted by  $A \setminus B$  or  $A - B$ , is defined as follows:

$$x \in A \setminus B \Leftrightarrow (x \in A \wedge x \notin B).$$

Why did we isolate this operator from the other three? Because this operator is not actually new—it can be derived from previous operators:  $A \setminus B = A \cap \overline{B}$ . We can intuitively justify this equivalence, but we will develop the ideas necessary to formally prove it a little later.

## 2.2 Venn Diagrams

Although Venn diagrams do *not* give formal proofs of set equivalences, they are often helpful when visualizing set operations.



If  $A$  and  $B$  are sets, then the circle on the left represents  $A$  and the circle on the right represents  $B$ . This diagram contains 4 disjoint regions, representing elements exclusively in  $A$ , exclusively in  $B$ , in both, or in neither. By using a Venn diagram, we can visualize many equivalences for sets, including the rules that we describe in the following section. We can also easily conjecture some equivalences for sets such as

$$B = (A \cap B) \cup (B \setminus A) \quad \text{and} \quad A \cap B = (A \cup B \setminus (A \setminus B)) \setminus (B \setminus A).$$

A similar diagram can be drawn for 3 sets, and such a diagram would have 8 disjoint regions. Notice that this has a combinatorial interpretation: for each set, an element can either be in or out of that set. Thus, there are  $2^3 = 8$  disjoint regions.

## 2.3 Rules for Sets

We've already seen two versions of De Morgan's laws: one for propositional logic, and one for predicate logic. Now we'll see the version of De Morgan's Laws for sets:

- The complement of the intersection is the union of the complements, i.e.,  $\overline{A \cap B} = \overline{A} \cup \overline{B}$ .
- The complement of the union is the intersection of the complements, i.e.,  $\overline{A \cup B} = \overline{A} \cap \overline{B}$ .

To prove these laws, we will (again) use De Morgan's laws for propositional logic. In general, to show that two sets  $X$  and  $Y$  are equal, we must prove that every element of  $X$  belongs to  $Y$  and vice versa. So we start by assuming  $x \in \overline{A \cap B}$  and apply rules:

$$\begin{aligned}
 x &\in \overline{A \cap B} \\
 &\Leftrightarrow x \notin A \cap B && \text{(definition of complement)} \\
 &\Leftrightarrow \neg(x \in A \cap B) && \text{(definition of } \notin) \\
 &\Leftrightarrow \neg(x \in A \wedge x \in B) && \text{(definition of } \cap) \\
 &\Leftrightarrow \neg(x \in A) \vee \neg(x \in B) && \text{(De Morgan's for } \wedge) \\
 &\Leftrightarrow (x \in \overline{A}) \vee (x \in \overline{B}) && \text{(definition of complement)} \\
 &\Leftrightarrow x \in \overline{A} \cup \overline{B}. && \text{(definition of } \cup)
 \end{aligned}$$

Since every line above is equivalent to the previous line, we have simultaneously shown both  $\overline{A \cap B} \subseteq (\overline{A} \cup \overline{B})$  and  $(\overline{A} \cup \overline{B}) \subseteq \overline{A \cap B}$ , as desired. Now we do the same thing for the other De

Morgan's law, by assuming  $x \in \overline{A \cup B}$  and applying rules:

$$\begin{aligned}
 x &\in \overline{A \cup B} \\
 &\Leftrightarrow x \notin A \cup B && \text{(definition of complement)} \\
 &\Leftrightarrow \neg(x \in A \cup B) && \text{(definition of } \notin) \\
 &\Leftrightarrow \neg(x \in A \vee x \in B) && \text{(definition of } \cup) \\
 &\Leftrightarrow \neg(x \in A) \wedge \neg(x \in B) && \text{(De Morgan's for } \vee) \\
 &\Leftrightarrow (x \in \overline{A}) \wedge (x \in \overline{B}) && \text{(definition of complement)} \\
 &\Leftrightarrow x \in \overline{A} \cap \overline{B}. && \text{(definition of } \cap)
 \end{aligned}$$

Now we consider other rules that we can apply on sets, all of which should be familiar:

- Commutativity:  $A \cup B = B \cup A$  and  $A \cap B = B \cap A$ .
- Associativity:  $(A \cup B) \cup C = A \cup (B \cup C)$  and  $(A \cap B) \cap C = A \cap (B \cap C)$ .
- Distributivity:  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$  and  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ .
- Idempotence:  $A \cap A = A$  and  $A \cup A = A$ .
- Complement:  $\overline{\overline{A}} = A$ .

Notice that if we think of  $\cup$  as  $\vee$  and  $\cap$  as  $\wedge$  (the symbols even resemble each other), then these rules are exactly the same as the rules for propositional logic. The proofs of all of these rules resemble the proofs given above for De Morgan's laws for sets, and verifying them is a good exercise.

Recall that propositional formulas have two normal forms known as DNF (" $\vee$  of  $\wedge$ 's") and CNF (" $\wedge$  of  $\vee$ 's"). Again, if we think of  $\cup$  as  $\vee$  and  $\cap$  as  $\wedge$ , then we can convert every set expression to DNF and CNF. Consider the following example:

$$X = \overline{A \cup (B \cap C)}.$$

We will apply a sequence of rules to convert  $X$  to a  $\cap$  of  $\cup$ 's:

$$\begin{aligned}
 X &= \overline{A \cup (B \cap C)} \\
 &= \overline{(A \cup B) \cap (A \cup C)} && \text{(distribute } \cup \text{ over } \cap) \\
 &= \overline{(A \cup B)} \cup \overline{(A \cup C)} && \text{(De Morgan's for } \cap) \\
 &= (\overline{A} \cap \overline{B}) \cup (\overline{A} \cap \overline{C}). && \text{(De Morgan's for } \cup)
 \end{aligned}$$

This final form is the  $\cup$  of  $\cap$ 's, so it is in "DNF for sets." Let's continue to convert this DNF to CNF, i.e., an intersection of unions. To make the rules easier to apply, we let  $R = (\overline{A} \cap \overline{B})$  and proceed:

$$\begin{aligned}
 X &= R \cup (\overline{A} \cap \overline{C}) \\
 &= (R \cup \overline{A}) \cap (R \cup \overline{C}) && \text{(distribute } \cup \text{ over } \cap) \\
 &= (\overline{A} \cup (\overline{A} \cap \overline{B})) \cap (\overline{C} \cup (\overline{A} \cap \overline{B})) && \text{(substitute } R, \text{ commutativity of } \cup) \\
 &= ((\overline{A} \cup \overline{A}) \cap (\overline{A} \cup \overline{B})) \cap ((\overline{C} \cup \overline{A}) \cap (\overline{C} \cup \overline{B})) && \text{(distribute } \cup \text{ over } \cap) \\
 &= (\overline{A} \cup \overline{A}) \cap (\overline{A} \cup \overline{B}) \cap (\overline{C} \cup \overline{A}) \cap (\overline{C} \cup \overline{B}) && \text{(associativity of } \cap)
 \end{aligned}$$

Note that at this point, we've written  $X$  as a  $\cap$  of  $\cup$ 's (i.e., "CNF for sets"), so technically, we are done. But we recall that  $\cup$  is idempotent, which implies  $\overline{A} \cup \overline{A} = \overline{A}$ . So our final version of  $X$  in "CNF for sets" is

$$X = \overline{A} \cap (\overline{A} \cup \overline{B}) \cap (\overline{C} \cup \overline{A}) \cap (\overline{C} \cup \overline{B}).$$

## 2.4 The Set Builder Notation

One way to describe a set is to explicitly write out its elements in curly brackets, such as  $S = \{1, 4, 3\}$  from above. However, this method is cumbersome when  $|S|$  is large, and completely fails when  $S$  is infinite. To combat this, and for the sake of clarity, we often use the set builder notation, which we now describe. Consider the following set:

$$A = \{x \in \mathbb{Z}^+ : \exists y \in \mathbb{Z}^+. x = 2y\}.$$

(Note: we sometimes use a "|" instead of a ":".) Here,  $A$  is the set of positive even integers. Notice that we are not explicitly stating which numbers belong to  $A$ , but rather, we give a predicate formula that determines whether a number belongs to  $A$ . Another example is the following, which defines a circle of radius  $r$ :

$$C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}.$$

You can probably guess what  $\mathbb{R}^2$  represents: it is the set of all points  $(x, y)$ , where  $x$  and  $y$  are real numbers. Note that unlike in sets, order matters here: the point  $(3, 5)$  is not equal to  $(5, 3)$ .

## 2.5 Common Sets

The following special symbols are used to denote common sets:

- $\mathbb{Z}$ : the set of integers
- $\mathbb{Q}$ : the set of rational numbers
- $\mathbb{R}$ : the set of real numbers
- $\mathbb{C}$ : the set of complex numbers

Notice that we have the hierarchy  $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ . A superscript  $+$  added to any of the above symbols (except  $\mathbb{C}$ ), restricts the set to its positive elements. For example,  $\mathbb{Z}^+$  denotes the set of positive integers. Similarly,  $\mathbb{Z}^-$  denotes the set of negative integers. If we want to refer to the set of non-negative integers, we can use  $\mathbb{Z}_0^+$ . Finally, the set of positive integers  $\mathbb{Z}^+$  is also known as the natural numbers, and this set is denoted by the symbol  $\mathbb{N}$ .

## 3 Summary

In this lecture, we introduced the notion of sets and basic operators on sets. We also saw Venn diagrams, rules (such as De Morgan's) for these operators, and the set builder notation. Finally, we took a look at notation used to refer to common sets.