Lecture 9

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1 Overview

In this lecture, we study a special class of relations on a set known as equivalence relations. We give examples and then prove a connection between equivalence relations and partitions of a set.

2 Equivalence Relations

Definition 1. *An* equivalence relation *is a relation that is reflexive, symmetric, and transitive.*

We shall now give some examples of equivalence relations. Recall that if R is a relation, then the statements "xRy", "$(x, y) \in R$", and "x relates to y" (but not "y relates to x") are all equivalent.

Example 1: Consider the relation $R = \{(x, y) : x = y\}$ on the set $\mathbb{Z}^+$. To show that R is an equivalence relation, we must show that R is reflexive, symmetric, and transitive:

- For all $x \in \mathbb{Z}^+$, the equality $x = x$ is true, so xRx. This proves R is reflexive.
- If xRy, then $x = y$, which means $y = x$, so yRx. This proves R is symmetric.
- If xRy and yRz, then $x = y$ and $y = z$, which means $x = z$, so xRz. This proves R is transitive.

Similarly, we can show that if S is any set, then the relation $\{(A, B) : A = B\}$ on the set 2^S is also an equivalence relation.

Example 2: Consider the sets $A = \{1, 2, a, b\}$ and $B = \{a, b\}$ (so $B \subset A$). Recall that if S is a set, then the power set of S is denoted by 2^S, and its elements are all possible subsets of S. Now consider the relation R on 2^A (i.e., R is a subset of $2^A \times 2^A$), defined as follows:

$$(X, Y) \in R \text{ if and only if } X \cap B = Y \cap B.$$

In other words, X relates to Y if they contain the same subset of letters. For instance, under this relation, $\emptyset$ relates to exactly four elements of 2^A: $\emptyset, \{1\}, \{2\}$, and $\{1, 2\}$.

We now show that R is an equivalence relation, which requires showing that R is reflexive, symmetric, and transitive. This proof (and many proofs of equivalence relations) is very similar to the proof in Example 1.

- For all $X \in 2^S$, we have $X \cap B = X \cap B$, so R is reflexive.
- If $(X, Y) \in R$ then $X \cap B = Y \cap B$, which means $Y \cap B = X \cap B$. This implies $(Y, X) \in R$, so R is symmetric.
- If $(X, Y) \in R$ and $(Y, Z) \in R$, then $X \cap B = Y \cap B$ and $Y \cap B = Z \cap B$. Together, these equalities imply $X \cap B = Z \cap B$, which means $(X, Z) \in R$, so R is transitive.

Example 3: Now let's slightly modify the definition of the relation from Example 2. Now, X relates to Y if and only if X and Y contain the same *number* of letters, that is, $|X \cap B| = |Y \cap B|$. It is straightforward to verify that this new relation is also an equivalence relation.

But note that this modified equivalence relation is indeed different from the one in Example 2. Consider $X = \{1, a\}$ and $Y = \{2, b\}$: here, X and Y share the same *number* of letters (one), but not the same *set* of letters ($\{a\}$ and $\{b\}$, respectively). Therefore, X does not relate to Y according to the relation defined in Example 2, but X relates to Y in the current example. In general, a set can have many different equivalence relations.

3 Partitions

One of the reasons that equivalence relations are interesting is because of their connection to partitions. To motivate this connection, let's consider the set $A = \{1, 2, 3\}$ and define a relation R on 2^A as follows: $(X, Y) \in R$ if and only if $|X| = |Y|$. We shall explicitly draw this relation below: for every element of 2^A, and if X relates to Y, we'll draw an arrow from X to Y.

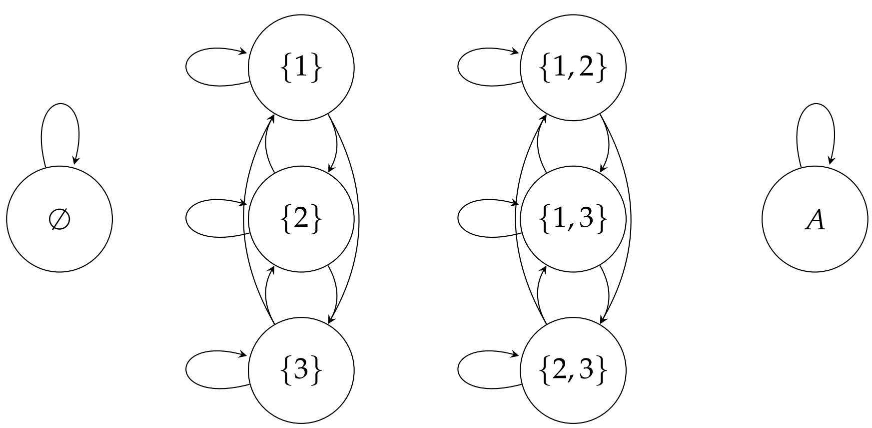

Figure 1: The equivalence relation $R = \{(X, Y) : |X| = |Y|\}$ on $A = \{1, 2, 3\}$, where an arrow from S_1 to S_2 indicates $(S_1, S_2) \in R$.

In Fig. 1, each circle represents an element of 2^A (i.e., a subset of A). Notice that based on the arrows, the elements of 2^A are grouped into four distinct clusters. Within each cluster, all of the elements relate to each other (including themselves, because R is reflexive), and if two elements are in different clusters, then neither relates to the other.

The structure of these clusters is no coincidence: if S is a set and R is an equivalence relation on S, then R *induces* a clustering of this form, and this kind of clustering is known as a partition.

Definition 2. *Let P be a set containing subsets of S, so P is a subset of 2^S. Then P is a* partition *of S if P satisfies the following properties:*

1. *The union of the elements of P is equal to S.*

2. *If S_1 and S_2 are elements of P and $S_1 \cap S_2 \neq \emptyset$, then $S_1 = S_2$.*

If P is a partition, then each element of P is called a block *of P, so each block of P is a subset of S.*

Informally, the first property ensures that every element of A is covered by a block, and the second property ensures that no two blocks overlap. An alternative definition is the following: if P is a collection of subsets of A, then P is a partition of A if every element of A is in exactly one subset in P. At the end of the lecture, we will formally state and prove the claim that every equivalence relation induces a partition, but for now, we continue exploring this idea via examples.

In Fig. 1, R induces the following partition on 2^A:

$$\{\emptyset\}, \Big\{\{1\}, \{2\}, \{3\}\Big\}, \Big\{\{1,2\}, \{1,3\}, \{2,3\}\Big\}, \{A\}.$$

This example may be somewhat confusing, because the elements of the set we partition (2^A) are actually subsets of some other set (A). Nonetheless, 2^A has 8 elements, and this partition assigns each element to blocks of size 1, 3, 3, and 1. We now give more examples that are more straightforward.

Example 1: Let $R = \{(a, b) : a = b\}$ be a relation on $\mathbb{Z}^+$; it's easy to verify that R is an equivalence relation. Now let P be the partition induced by R. Then for all $n \in \mathbb{Z}^+$, the only element to which n relates is n itself. Thus, every block of the partition contains exactly one positive integer, and there are infinitely many blocks.

Example 2: Let $R_2 = \{(a, b) : a \equiv b \pmod 2\}$ be a relation on $\mathbb{Z}$. (We say that $a \equiv b \pmod k$ if $a - b$ is divisible by k.) Again, it's easy to verify that R is an equivalence relation. Notice that if $(x, y) \in R$, then $x - y$ is even, which means x and y are either both even or both odd. Thus, the partition induced by R_2 has two blocks: the set of even integers, and the set of odd integers.

Similarly, if $R_{16} = \{(a, b) : a \equiv b \pmod{16}\}$ is a relation on $\mathbb{Z}$, then R_{16} is also an equivalence relation, so R_{16} induces a partition. The elements of each block all have the same remainder after dividing by 16, and there are 16 possible remainders $(0, 1, \ldots, 15)$, so this partition has 16 blocks.

To begin formalizing the connection between equivalence relations and partitions, we now state a useful definition regarding equivalence relations.

Definition 3. *Suppose R is an equivalence relation on a set S, and x is any element of S. The* equivalence class *of x, denoted by $[x]$, is the set $\{y \in S : xRy\}$.*

Example 1: In Fig. 1, recall that 2^A is being partitioned, and 2^A has 8 elements. The corresponding equivalence classes are the following:

$$\begin{aligned}
[\emptyset] &= \{\emptyset\} \\
[\{1\}] = [\{2\}] = [\{3\}] &= \Big\{\{1\}, \{2\}, \{3\}\Big\} \\
[\{1,2\}] = [\{1,3\}] = [\{2,3\}] &= \Big\{\{1,2\}, \{1,3\}, \{2,3\}\Big\} \\
[A] &= \{A\}.
\end{aligned}$$

Notice that the equivalence classes form a partition—this is the connection between equivalence relations and partitions that we will prove below.

Example 2: Recall the relation $R_{16} = \{(a,b) : a \equiv b \pmod{16}\}$ on $\mathbb{Z}$. Some of the equivalences classes are given below:

$$\begin{aligned}
[0] &= \{\ldots, -32, -16, 0, 16, 32, \ldots\}, \\
[1] &= \{\ldots, -31, -15, 1, 17, 33, \ldots\}, \\
&\vdots \\
[15] &= \{\ldots, -33, -17, -1, 15, 31 \ldots\}.
\end{aligned}$$

We are now ready to state and prove a theorem that formally describes the fundamental relationship between equivalence relations and partitions. Note that throughout this lecture, we have already seen that an equivalence relation induces a partition, but now we shall formally prove this phenomenon.

Theorem 1. *If R is an equivalence relation on a set S, then the equivalence classes of R partition S.*

Proof. Let P denote the subsets described in the theorem, so $P = \{[x] : x \in S\}$. Recall that P is a partition if the following properties hold:

1. The union of the elements of P is equal to S.

2. If S_1 and S_2 are elements of P and $S_1 \cap S_2 \neq \emptyset$, then $S_1 = S_2$.

Let's begin with the first property: let s be any element of S. Since R is reflexive, we have sRs, which means $s \in [s]$. Since $[s]$ is an element of P, s is covered by an element of P, so P satisfies the first property.

To prove the second property, we start by letting S_1 and S_2 be elements of P such that $S_1 \cap S_2 \neq \emptyset$. By the definition of P, there exist a, b such that $S_1 = [a]$ and $S_2 = [b]$. Since $S_1 \cap S_2 \neq \emptyset$, there exists z such that $z \in [a] \cap [b]$. Thus, aRz and bRz. Since R is symmetric, this means zRa and zRb. Furthermore, since R is transitive, this means aRb and bRa.

We will now prove that $[a] = [b]$, and as usual, we begin by showing $[a] \subseteq [b]$: let x be any element of $[a]$. Then aRx, and combined with bRa and transitivity of R, we have bRx. Thus, $x \in [b]$, so $[a] \subseteq [b]$.

Similarly, let y be any element of $[b]$. Then bRy, and combined with aRb and transitivity of R, this means aRy. Thus, $y \in [a]$, so $[b] \subseteq [a]$. Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we must have $[a] = [b]$, as desired. $\square$

4 Summary

In this lecture, we learned about a special class of relations known as equivalence relations. We saw multiple examples, and ended by proving that the equivalence classes of an equivalence relation form a partition of the set that the relation is defined on.