1. In this problem we consider the 0-1 knapsack problem: Given \( n \) items, with item \( i \) being worth \( v[i] \) dollars and having weight \( w[i] \) pounds, fill a knapsack of capacity \( m \) pounds with the maximal possible value.

**Example:** Given a knapsack of capacity 50, the maximal value obtainable with three items of value $60, $100, and $120 and weights 10, 20, and 30, respectively, is $220.

The algorithm \texttt{Knapsack}(i,j) below returns the maximal value obtainable when filling a knapsack of capacity \( j \) using items among items 1 through \( i \) (\texttt{Knapsack}(n,m) solves our problem). The algorithm works by recursively computing the best solution obtainable \textit{with} the last item and the best solution obtainable \textit{without} the last item, and returning the best of them.

\texttt{Knapsack}(i,j)

\begin{verbatim}
    IF w[i] <= j THEN
        with = v[i] + Knapsack(i-1, j-w[i])
    ELSE
        with = 0
    END IF
    without = Knapsack(i-1,j)
    RETURN max{with, without}
\end{verbatim}

(a) Show that the running time \( T \) of \texttt{Knapsack}(n,m) is exponential in \( n \) or \( m \). (Hint: look at the case where \( w[i] = 1 \) for all \( 1 \leq i \leq n \) and show that \( T(n,m) = \Omega(2^{\min(m,n)}) \)).
(b) Describe an $O(n \cdot m)$ algorithm for computing the value of the optimal solution.

Solution:

(a) Following the hint, if $w[i] = 1$ then it is clear that $T(n, m) > 2T(n - 1, m - 1) + 1$.
This recurrence, which runs for $\min(m, n)$ steps, gives that $T(n, m) = \Omega(2^{\min(m, n)})$.

(b) We create a table of size $[n][m]$ in which to store our results of prior runs. The modified algorithm would be as follows:

```plaintext
Knapsack(i, j)

IF table[i][j] != 0 THEN
    RETURN table[i][j]
IF w[i] <= j THEN
    with = v[i] + Knapsack(i-1, j-w[i])
ELSE
    with = 0
without = Knapsack(i-1, j)
```

```plaintext
table[i][j] = max{with, without}
RETURN max{with, without}
```

END

This will run in $O(n \cdot m)$ time as we fill each entry in the table at most once, and there are $nm$ spaces in the table.