CPS 130 Homework 4 - Solutions

1. (CLRS 4.2-2) Argue that the solution to the recurrence

   \[ T(n) = T(n/3) + T(2n/3) + n \]

   is \( \Omega(n \log n) \) by appealing to a recursion tree.

   **Solution:** The recursion tree for the recurrence is given in CLRS. Note that the values across the levels of the recursion tree sum to \( n \). The shortest path from a root to a leaf is

   \[ n \rightarrow \frac{1}{3}n \rightarrow \left(\frac{1}{3}\right)^2 n \rightarrow \ldots \rightarrow 1. \]

   Since \( (1/3)^k n = 1 \) when \( k = \log_3 n \), the height of the tree is at least \( \log_3 n \). Thus the solution to the recurrence is at least \( n \log_3 n = \Omega(n \log n) \).

2. Give asymptotic upper and lower bounds for the following recurrences. Assume \( T(n) \) is constant for \( n \leq 2 \). Make your bounds as tight as possible, and justify your answers.

   - \( T(n) = T(n - 1) + n \)
     **Solution:** Note that you cannot apply Master method. Assume \( T(1) = 1 \) and iterate:

     \[
     T(n) &= T(n - 1) + n \\
     &= T(n - 2) + (n - 1) + n \\
     &= \ldots \\
     &= T(n - 1) + (n - 1 + 1) + \ldots + n \\
     &= T(1) + 2 + 3 + \ldots n \\
     &= \sum_{i=1}^{n} i \\
     &= \frac{n(n+1)}{2} \\
     &= \Theta(n^2)
     \]

   - \( T(n) = T(\sqrt{n}) + 1 \)
     **Solution:** Note that you cannot apply Master method. Assume \( T(2) = 1 \) and iterate:

     \[
     T(n) &= T(n^{1/2}) + 1 \\
     &= T(n^{1/2}) + 1 + 1 \\
     &= \ldots \\
     &= T(n^{1/2}) + i
     \]
For this type of recurrence we cannot stop at 1 because \( \frac{1}{\log n} \) cannot be 1, except at the limit. The recursion depth is given by \( \frac{1}{\log n} = 2 \), which gives \( i = \log \log n \).

Substituting in the relation above we get:

\[
T(n) = T(2) + \log \log n
\]

\[
= \Theta(\log \log n)
\]

- \( T(n) = 2T(n/2) + n/\log n \)

**Solution:** Note that you cannot apply Master method. Assume \( T(1) = 1 \) and iterate:

\[
T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log n}
\]

\[
= 2 \cdot 2T\left(\frac{n}{2^2}\right) + 2 \cdot \frac{n}{\log \frac{n}{2}}
\]

\[
= ... 
\]

\[
= 2^iT\left(\frac{n}{2^i}\right) + \sum_{k=0}^{i-1} \frac{n}{\log \frac{n}{2^k}}
\]

The recursion depth is \( i = \log_2 n \) and substituting we get:

\[
T(n) = 2^{\log_2 n} T(1) + \sum_{k=0}^{\log_2 n - 1} \frac{n}{\log n - k}
\]

\[
= nT(1) + n \cdot \sum_{k=0}^{\log_2 n - 1} \frac{1}{\log n - k}
\]

\[
= nT(1) + n \cdot \sum_{k=1}^{\log_2 n} \frac{1}{k}
\]

\[
= nT(1) + n \cdot H_{\log n}
\]

\[
= nT(1) + n \cdot \Theta(\log \log n)
\]

\[
= \Theta(n \log \log n)
\]

Here we used the standard notation for the harmonic sum

\[
H_n = \sum_{k=1}^{n} \frac{1}{k} = \Theta(\log n)
\]

- \( T(n) = T(n - 1) + 1/n \)

**Solution:** Assume \( T(1) = 1 \) and iterate:

\[
T(n) = T(n - 1) + \frac{1}{n}
\]

\[
= T(n - 2) + \frac{1}{n - 1} + \frac{1}{n}
\]

\[
= ... 
\]

\[
= T(n - i) + \frac{1}{n - i + 1} + \ldots + \frac{1}{n}
\]
Taking $i = n - 1$ we get:

\[
T(n) = T(1) + \frac{1}{2} + \ldots + \frac{1}{n}
\]
\[
= \sum_{k=1}^{n} \frac{1}{k}
\]
\[
= H_n = \Theta(\log n)
\]