Theorem: \( \mathbb{R} \) is uncountable.

Proof: By diagonalization to obtain a contradiction.

- Goal is to show that there is no one-to-one correspondence between \( \mathbb{N} \) and \( \mathbb{R} \).
- We'll show this by contradiction.

- Without loss of generality, we only consider \([0,1]\) and for the sake of contradiction, let \( f \) be a bijection \( f: \mathbb{N} \to [0,1] \).
- Then we should be able to list all the numbers in \([0,1]\).
- Let's list \( A \) as infinite arrays (2D) as below:

\[
A = \begin{cases} 
  f(1) = 0 \cdot a_{11}, a_{12}, a_{13}, a_{14}, \ldots \\
  f(2) = 0 \cdot a_{21}, a_{22}, a_{23}, a_{24}, a_{25}, \ldots \\
  f(3) = 0 \cdot a_{31}, a_{32}, a_{33}, a_{34}, a_{35}, \ldots \\
  f(4) = 0 \cdot a_{41}, a_{42}, a_{43}, a_{44}, a_{45}, \ldots \\
  f(5) = 0 \cdot a_{51}, a_{52}, a_{53}, a_{54}, a_{55}, \ldots \\
  \vdots \end{cases}
\]
• If we can show that given A, we can find a number $x \in (0,1)$ s.t. it is not in A, we will reach a contradiction.

• Let $x = 0.\overline{d}_1 d_2 d_3 d_4 d_5 \cdots$.

• The $d_i$'s are constructed as follows:
\[ d_n = \begin{cases} a_{n+1} + 1 & \text{if } a_{n} \in \{0,1,2,\ldots,8\} \\ 8 & \text{if } a_{n} = 9. \end{cases} \]

• Note: $d_n \neq a_n$.

• Claim: $\forall n \in \mathbb{N}$, $f(n) \neq x$, because it differs from $f(n)$ in the $n$-th diagonal digit.

• Note: $n$-th digit of $f(n) = a_n$, and $n$-th digit of $x = d_n$. They are different by construction.

So $a_n \neq d_n$ by construction.

• Therefore, we reached a contradiction. Hence $\mathbb{R}$ is uncountable.
Example:

\[ n \rightarrow \frac{f(n)}{f(0)} \]

1 \quad 0.14662\ldots
2 \quad 0.39471\ldots
3 \quad 0.48247\ldots
4 \quad 0.21432\ldots
5 \quad 0.81243\ldots

\[ x = 0.19233\ldots \]

Clearly, \( x \) cannot be any of the above five elements of the list. In fact, it cannot be ANY element in the list!

Exercise: The power set of \( \mathbb{N} \) i.e. \( \mathcal{P}(\mathbb{N}) \) is uncountable. Stated otherwise, the set of infinite sequences of 0's and 1's is uncountable.