1. (16 pts) Complete or answer the following. If the resulting answer has less than 10 items, just list the items. Otherwise give a formal description of the language.

\[ L_1 = \{b, abc\}, \Sigma = \{a, b, c\} \]
\[ L_2 = \{a, ab, abc\}, \Sigma = \{a, b, c\} \]
\[ L_3 = (ab)^*(ba)^*, \Sigma = \{a, b\} \]
\[ L_4 = \{w \in \Sigma^* | n_a(w) = n_b(w)\}, \Sigma = \{a, b\} \]
\[ L_5 = \{w \in \Sigma^* | n_a(w) \text{ is even}\}, \Sigma = \{a, b\} \]

(a) \( L_1 \times \{c\} = \{b, c\}, \{abc, c\}\)

(b) \(2^{L_1} = \{\emptyset, \{\emptyset\}, \{b\}, \{b, \{\emptyset\}, \{abc\}, \{b, abc\}\}\}

(c) \(|L_2| = 3\)

(d) \(L_1 \cup L_2 = \{a, b, ab, abc\}\)

(e) \(L_1 \cap L_2 = \{abc\}\)

(f) \(L_3 \cap L_4 = L_3\)

(g) \(L_5 = \{w \in \Sigma^* | n_a(w) \text{ is odd}\}\)

(h) \(L_1 \circ L_3 = (b + abc)(ab)^*(ba)^*, \Sigma = \{a, b, c\}\)

2. (20 pts) Answer TRUE or FALSE to each of the statements below.

(a) \(\{a\} \in \{a, b, c, d\}\)? (TRUE or FALSE?)

(b) If \(L_1\) is a regular language and \(L_2 = \{w \in \Sigma^* | n_a(w) \text{ is even}\}, \Sigma = \{a, b\}\), then \(L_1 \cap L_2\) is a regular language. (TRUE or FALSE?)

(c) If \(L\) is a regular grammar with only one variable, then \(L\) is a context-free grammar. (TRUE or FALSE?)
(d) If $M$ is an NPDA with only two states, then there exists an NFA $M'$ such that $L(M) = L(M')$. (TRUE or FALSE?)

(e) If $G$ is a CFG with only one production, then there exists a regular expression $E$ such that $L(G) = L(E)$. (TRUE or FALSE?)

(f) If $G$ is an unambiguous CFG then there exists an NFA $M$ such that $L(G) = L(M)$. (TRUE or FALSE?)

(g) $L = \{ w \in \Sigma^* \mid n_a(w) > n_b(w) > n_c(w) \}$, $\Sigma = \{a, b, c\}$. $L$ is regular. (TRUE or FALSE?)

(h) $L = \{ a^n b^m c^k \mid k > n, k > m, k > 50, n > 10, m > 5 \}$, $\Sigma = \{a, b, c\}$. $L$ is regular. (TRUE or FALSE?)

(i) $L = \{ w \in \Sigma^* \mid \text{ba and abbb are not substrings in } w \}$, $\Sigma = \{a, b, c\}$. $L$ is regular. (TRUE or FALSE?)

(j) $L = \{ a^n b^{2n} \mid n \text{ is even, } n > 0 \}$, $\Sigma = \{a, b\}$. $L$ is regular. (TRUE or FALSE?)

3. (3 pts) Let $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$ be a NPDA. Explain what the following statement means:

$L(M) = \{ w \in \Sigma^* \mid (q_0, w, z) \xrightarrow{*} (p, \lambda, u), p \in F, u \in \Gamma^* \}$.

The strings accepted by NPDA $M$ are those strings from $\Sigma^*$ such that if starting with state $q_0$, input string $w$ on the tape, and $z$ on stack, then state $p$ (a final state) is reached, all symbols in $w$ have been processed, and there are no symbols on the stack.
4. (8 pts) Draw a DFA for the following language. You do not have to show trap states (show the transition diagram, indicate the start state by a short arrow, and final states by double circles.)

$L = \{ w \in \Sigma^* \mid n_a(w) \bmod 4 = 2 \text{ and } \text{bb is not a substring of } w \}$, $\Sigma = \{a, b\}$.

For example, $baba$, $aaabaaa$ and $baab$ are in $L$. 
5. \[ S \rightarrow B a b | S a | \lambda \\
B \rightarrow B b | S \]

Language is \((b^* a b + a)^*\)
6. (6 pts) Write a context-free grammar for the following language.

\[ L = \{a^n b^n b^m d^p \mid p > m, m > 0, n \geq 0\} \]

\[ S \rightarrow AB \\
A \rightarrow aAb \mid \lambda \\
B \rightarrow bBd \mid Bd \mid bdd \]
7. (6 pts) Consider the following DFA.

a) Show states $q_1$ and $q_2$ are distinguishable with an appropriate string. Explain.
$$q_1(ab) \rightarrow q_4 \text{ non-final} \quad q_2 \text{ on ab} \rightarrow q_5 \text{ final}$$

b) Give the states in the minimal state DFA (you do not need to show the arcs). Each state should indicate which states it represents from the original DFA. For example you could list one state as 0,1,2 if states 0, 1 and 2 in the original DFA can be combined to form a state in the minimal state DFA.
8. (10 pts) Consider $L = \{a^n c^m d^p \mid p > m, n \geq 0, m > 0\}$. Draw the transition diagram for a nondeterministic pushdown automaton $M$ that accepts $L$ by final state. (Remember to identify the start state by an arrow and final states by double circles. Format of labels are $a, b; cd$ where $a$ is the symbol on the tape, $b$ is the symbol on top of the stack that is popped, and $cd$ are pushed onto the stack (with $c$ on top of $d$). $Z$ is on top of the stack when $M$ starts.)

(a) First list 3 strings in $L$. $bddd, acbdd, acbddd$

(b) Now draw the transition diagram.

[Diagram image]
9. (6 pts) **Pumping Lemma:** Let \( L \) be an infinite regular language. \( \exists \) a constant \( m > 0 \) such that any \( w \in L \) with \( |w| \geq m \) can be decomposed into three parts as \( w = xyz \) with

\[
\begin{align*}
|xy| &\leq m \\
|y| &\geq 1 \\
x y^i z &\in L \quad \text{for all } i \geq 0
\end{align*}
\]

Use the Pumping Lemma to prove the language \( L \) below is not regular.

\( L = \{ w \in \Sigma^* | n_a(w) + n_b(w) > n_c(w) \} \Sigma = \{a, b, c\} \)

For example, \( aabac \) and \( ccbaba \in L \).

**Proof:** (SHOW ALL STEPS! Some have been started for you.)

Assume \( L \) is regular

Choose \( w = a^m b c^m \)

Show there is no way to partition this string \( w = xyz \) such that the properties of the pumping lemma hold.

\[
y = a^j \quad j > 0 \\
x = a^k \quad k \geq 0 \quad |xy| \leq m \\
z = a^{m-j-k} b c^m
\]

\[
xy^0z \in L \\
= a^{m-j} b c^m \notin L \quad \text{since } \quad n(a) + n(b) \leq n(c)
\]

\( \Rightarrow \) contradiction!

\( \Rightarrow L \) is not regular
10. (8 pts) Consider the following property, SwapFirstAWithLastB. If $L$ is a regular language, then $\text{SwapFirstAWithLastB}(L)$= strings from $L$ that have the first $a$ replaced with $b$ and the last $b$ replaced with $a$. If there is a string $w$ that does not have at least one $a$ and one $b$ in the string, then $w$ is NOT in $\text{SwapFirstAWithLastB}$.

For example, if $caab$ is in $L$, then $chaba$ is in $\text{SwapFirstAWithLastB}(L)$. If $bbaab$ is in $L$, then $bbbbaa$ is in $\text{SwapFirstAWithLastB}(L)$. If $ba$ is in $L$, then $ab$ is in $\text{SwapFirstAWithLastB}(L)$.

If $ab$ is in $L$, then $ab$ does not generate a string in $\text{SwapFirstAWithLastB}(L)$.

Prove that $\text{SwapFirstAWithLastB}(L)$ is a regular language.

Idea: $L$ is regular. $M$ is a DFA for $L$.
Make 4 copies of $M$ called $M_1, M_2, M_3, M_4$.

Explanation →
Changes:

1) M₁, delete all a arcs and replace with a b arc to corresponding cousin in M₂.
2) M₁, for each b arc add an arc labeled a to corresponding cousin state in M₃.
3) M₂, for each b arc add an arc labeled a to corresponding cousin state in M₄.
4) M₃ remove all a arcs and replace with b arc to corresponding cousin state in M₄.
5) Start in M₁.
6) Final states in M₁, M₂ and M₃ are now non-final.
7) Remove b arcs in M₃ and M₄.

The first a is replaced and you move into M₂. The last b is replaced and you move into M₄ with no more b's to process. If the first a comes after the last b as in ba, you replace the last b and go into M₃, then first a and into M₄.