1. (14 pts) Complete or answer the following. If the resulting answer has more than 4 items, just list 4 items unless it otherwise states the number of items to list.

\[ A = \{10, 20, 30, 40, \ldots\} \quad B = \{3, 6, 9, 12, \ldots\} \quad C = \{3, 4, 8\} \quad D = \{2, 6, 8, 9\} \]

(a) \[ A \cap B = \]
(b) \[ B \times B = \]
(c) \[ | C \times D | = \]
(d) In the proof of converting an NPDA to a CFG, variables are defined as \((q_i c q_j)\), which means in going from state \(q_i\) to state \(q_j\) the stack changes how?

(e) If \(M\) is an NFA with \(k\) states, then the equivalent DFA will have \underline{________________________} or fewer states. (fill in the blank)

(f) Let \(A = \{a, b, c, d\}\). List 6 strings in \(A^*\).

(g) Let \(A = \{a, b, c, d\}\), \(B = \{a, bc\}\). List 6 strings in \(A \circ B \circ B\).

2. (18 pts) Answer TRUE or FALSE to each of the statements below.

(a) \(\emptyset \in \emptyset \times \emptyset\) (TRUE or FALSE?)

(b) If \(M\) is a DFA that has more than one final state, then there exists a DFA \(M'\) with exactly one final state such that \(L(M) = L(M')\). (TRUE or FALSE?)

(c) If \(M\) is an NFA with 100 states, then there exists a minimal state DFA \(M'\) with 50 or fewer states such that \(L(M) = L(M')\). (TRUE or FALSE?)

(d) The two regular expressions \(a(b + \lambda)^*\emptyset + a\) and \(ab^*\) are equivalent. (TRUE or FALSE?)

(e) \(L = \{a^n b^{2m} \mid n + m \text{ is even, } n > 0, m > 0\}\). L is regular. (TRUE or FALSE?)

(f) \(L = \{w \in \Sigma^* \mid n_a(w) = n_b(w) + 1\}, \Sigma = \{a, b\}\). L is regular. (TRUE or FALSE?)

(g) \(L = \{w \in \Sigma^* \mid n_a(w) = n_b(w) + 1 \text{ and } n_a(w) < 100\}, \Sigma = \{a, b\}\). L is regular. (TRUE or FALSE?)

(h) \(L = \{w \in \Sigma^* \mid n_a(w) = n_b(w) + n_c(w), n_b(w) > 100, \text{ and } n_c(w) < 50\}, \Sigma = \{a, b, c\}\). L is regular. (TRUE or FALSE?)

(i) If \(R\) is a regular expression, then there exists an NPDA \(M\) such that \(L(R) = L(M)\). (TRUE or FALSE?)
3. (8 pts) Draw a DFA for the following language. Do not show trap states. (show the transition diagram, indicate the start state by a short arrow, and final states by double circles.)

\[ L = \{ w \in \Sigma^* \mid n_a(w) \mod 3 = 1 \text{ and } n_b(w) \mod 3 = 2 \} \quad \Sigma = \{ a, b \} \]

For example, \( bba, aabbaa \) and \( abababbb \) are in \( L \).

4. (5 pts) Consider the following DFA with \( \Sigma = \{ a, b \} \). Give a regular grammar that represents the same language.

\[
\begin{array}{c}
q_0 \xrightarrow{a} q_1 \\
q_1 \xrightarrow{b} q_0
\end{array}
\]

Give the equivalent regular grammar here.

5. (5 pts) Give a regular expression for the following language.

\[ L = \{ w \in \Sigma^* \mid n_a(w) \mod 3 = 0 \text{ and there is exactly } 1 \text{ } b \} \quad \Sigma = \{ a, b \} \]

For example, \( abaa \) and \( baaa \) are in \( L \).

Pumping Lemma: Let \( L \) be an infinite regular language. \( \exists \) a constant \( m > 0 \) such that any \( w \in L \) with \( |w| \geq m \) can be decomposed into three parts as \( w = xyz \) with

\[
\begin{align*}
|xy| &\leq m \\
|y| &\geq 1 \\
x^{i}y^{i}z &\in L \quad \text{for all } i \geq 0
\end{align*}
\]

6. (6 pts) Use the Pumping Lemma to prove

\[ \Sigma = \{ a, b \}, L = \{ a^n b^p \mid 0 < p < 2n \} \text{ is not regular.} \]

Proof: (SHOW ALL STEPS! Some have been started for you.)

Assume \( \text{_________________________________________} \)

Choose \( w = \text{_________________________________________} \)

7. (3 pts) Consider the following proof to prove the language \( L \) is not regular. \( \Sigma = \{ a, b, c \}, L = \{ a^n b^m c^p \mid 2m = n + p, m > 0, n > 0, p > 0 \} \)

What is the first incorrect statement in the proof? Explain why.

- Proof:
  
  Assume \( L \) is regular.
Define the homomorphism $h(L)$ by $h(a) = a, h(b) = bb$ and $h(c) = a$.
$L_2 = h(L) = \{a^nb^m a^p \mid m = n + p, m > 0, n > 0, p > 0\}$, with $\Sigma = \{a, b\}$. $L_2$ is regular by closure of homomorphism.
$L_3 = \{a^n \mid n > 0\}$, $L_3$ is regular.
$L_4 = L_2 - L_3 = \{a^nb^m \mid n > 0, m > n\}$, $L_4$ is regular by closure of difference.
$L_5 = L_3 \circ L_4 = \{a^{n+p}b^m \mid n > 0, p > 0, m = n + p\} = \{a^nb^p \mid n > 0\}$ , $L_5$ is regular by closure of concatenation
Contradiction! already shown $L_5$ is not regular.
Thus, $L$ is not regular. QED.

8. (10 pts) Consider $L = \{a^n b^m \mid m > 0, m < 2n \text{ and } m \text{ is odd}\}$. Draw the transition diagram for a nondeterministic pushdown automaton $M$ that accepts $L$ by final state. 
(Remember to identify the start state by an arrow and final states by double circles. Format of labels are $a,b;cd$ where $a$ is the symbol on the tape, $b$ is the symbol on top of the stack that is popped, and $cd$ are pushed onto the stack (with $c$ on top of $d$). $Z$ is on top of the stack when $M$ starts. ).

(a) First list 3 strings in $L$.

(b) Now draw the transition diagram.

9. (8 points) Write a context-free grammar for the following languages.

(a) $L = \{a^{2n}b^n \mid n > 0\} \Sigma = \{a, b\}$.

(b) $L = \{w \in \Sigma^* \mid n_a(w) > 2 * n_b(w)\} \Sigma = \{a, b\}$.

10. (8 pts) Consider the following property, Replace_Last_bs_With_as (RLbsWas). If $L$ is a regular language, then

$$RLbsWas(L) = \{w = uy \mid uv \in L, u \in \Sigma^*, v, y \in \Sigma^+, v \text{ has at least one } b, y \text{ is the same as } v \text{ but with every } b \text{ replaced with an } a\}, \Sigma = \{a, b\}$$

In other words, $RLbsWas(L)$ accepts those words from $L$ with one or more $b$’s each replaced with an $a$, and for any $b$ that is replaced, all $b$’s to its right in the string must also be replaced. Note that if a string $w \in L$ does not have a $b$, then the string is not in $RLbsWas(L)$.

For example, consider the simple language $L$ with just one string, $L = \{ababbb\}$. Then the corresponding strings in $RLbsWas(L)$ are $ababba$ (rightmost $b$ replaced) , $ababaa$ (rightmost 2 $b$’s replaced), $abaaaaa$ (rightmost 3 $b$’s replaced) and $aaaaaaa$ (all $b$’s replaced). Note that $ababab$ is not in $RLbsWas$ since a $b$ was replaced by an $a$, but there is a $b$ to its right that was not replaced by an $a$.

Prove that the regular languages are closed under the $RLbsWas(L)$ property. (Show all steps! A picture may be helpful but you must explain it.)