# Symbolic and Numerical Computation for Artificial Intelligence

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# Chapter 6

# 2D and 3D Object Recognition and Positioning with Algebraic Invariants and Covariants

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We describe part of our model-based approach to 3D rigid object recognition and positioning from range data: methods for fast classification and positioning of algebraic surfaces. Since we are primarily interested in recognizing and positioning rigid objects, we focus on methods invariant under Euclidean coordinate transformations, but we also include some extensions to the affine and projective cases. These cases are related to other recognition and positioning problems involving 2D or 3D curves. Our approach to model-based object recognition and positioning can be divided in four stages. In the first stage algebraic surfaces are fitted to regions of the data set. Then, in the second stage, a data base is searched for regions of known objects with algebraic surface approximations similar to those fitted to the data regions. In the third stage, a matching coordinate transformation is computed for each matching candidate extracted from the data base. These coordinate transformations are computed using explicit formulas, and constitute the estimated positions for the initially recognized object. These matches constitute initial hypotheses for the presence of the associated objects in the data regions. Finally, the hypotheses are globally tested to sort out inconsistencies. We start by reviewing some of our previous work on computationally attractive fitting of algebraic curves and surfaces. Then we introduce methods to solve the classification and positioning problems, the second and third stages of our approach. We base the fast data base search for similar algebraic surfaces, the classification problem, on comparing algebraic invariants. We compute the matching coordinate transformation between two surfaces with approximately the same invariants by associating an intrinsic frame of reference to every algebraic surface. This frame of reference, an object-based coordinate system, is a covariant function of the coefficients of the polynomial which defines the surface, and its relative location with respect to the surface is independent of the viewpoint. For a nonsingular quadratic surface, the eigenvalues of the matrix associated with the second degree terms of the defining polynomial are Euclidean invariants. The center of symmetry of the surface, and the eigenvectors of the same matrix define an intrinsic frame of reference. The methods introduced in this paper generalize these constructions to algebraic curves and surfaces of higher degree.

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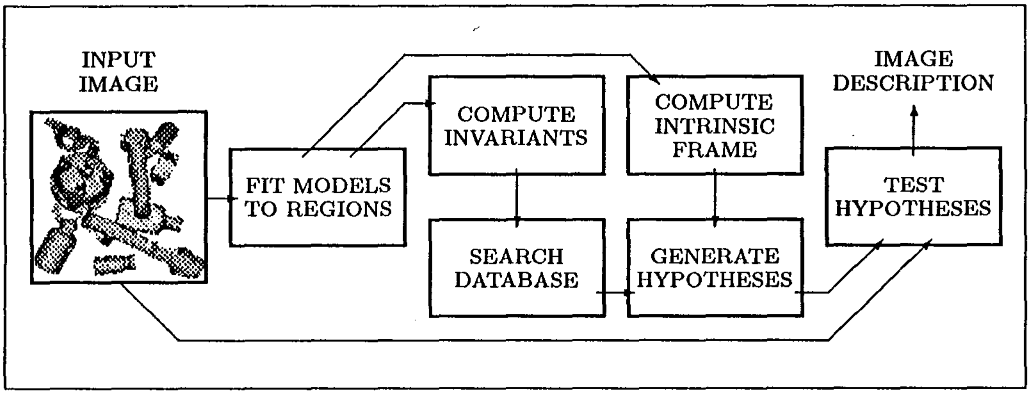


**Figure 1.** Global structure of the object recognition and positioning system.

## 1. Introduction

We describe part of our model-based approach to 3D rigid object recognition and positioning from range data: methods for fast classification and positioning of algebraic surfaces. Since we are primarily interested in recognizing and positioning rigid objects whose boundary surfaces can be well approximated by piecewise algebraic surfaces, sets of zeros of polynomials in three variables, we will focus on Euclidean invariant methods in 3D. However, we also include some extensions to affine and projective invariant methods, which are related to other recognition and positioning problems involving 2D or 3D curves.

Figure 1 describes the global structure of our approach. Due to the problems of occlusion, known solid objects are represented in a database as hierarchical collections of regions of boundary surfaces, of different sizes. Models, in our case algebraic surfaces, are fitted to small regions of the data set. These regions are small enough so that most of them correspond to a single object, but big enough to contain sufficient information to uniquely determine the location and orientation of an object. Alternatively, models are fitted to smaller regions, and symbolic methods are used to compute the parameters of the model which fits a group of these small regions. Since the parameters of these models are coordinate system dependent, a vector of *invariants* is computed for each model. An *invariant* is a function of the parameters which yields the same values independently of the viewer coordinate system. This vector of invariants is used to index into the database of regions of known models. Using these invariants, the database can be organized for an efficient search. The database search produces a list of triples. Each of these triples consists of a region of a known object, the object that the region corresponds to, and the coordinate transformation from the object coordinate system to the region coordinate system. If one or more matches are found in the database, the *intrinsic coordinate system* of the data set region is computed and, for every match found, the coordinate transformation which best aligns the data region with the model region is computed using the intrinsic coordinate systems of the two matching regions. This coordinate transformation constitutes a hypothesis corresponding to the presence of the associated object in the computed position and orientation. The hypotheses generated in this way are then globally tested, and the final interpretation of the data set is produced.

With the notation to be introduced later in the paper, if $F_{[1,1]}$ and $G_{[1,1]}$ are nonsingular symmetric $3\times 3$ matrices, $F_{[1]}$ and $G_{[1]}$ are three dimensional vectors, and $F_{[0]}$ and $G_{[0]}$ are constants, the second degree polynomials

$$\begin{aligned} f(x) &= \tfrac{1}{2}x^t F_{[1,1]}x + F_{[1]}^t x + F_{[0]} \\ g(x) &= \tfrac{1}{2}x^t G_{[1,1]}x + G_{[1]}^t x + G_{[0]} \end{aligned}$$

in three variables $x = (x_1, x_2, x_3)^t$ define two quadric surfaces. It is well known that a necessary condition (not sufficient) for these two quadric surfaces to be congruent, i.e. for the existence of an Euclidean transformation $x' = T(x)$ such that $f(T(x)) = g(x)$, is that the three eigenvalues of the two matrices coincide, except for a common multiplicative factor. These eigenvalues are Euclidean invariants of the surfaces. If that is the case, we can obtain the best matching coordinate transformation from the corresponding intrinsic frames of reference. The center of a nonsingular quadric surface, the origin of its intrinsic frame of reference, is the unique point in space which, if taken as the origin of the coordinate system, makes the polynomial defining the surface have zero linear part. That is, the center of the quadric surface defined by the polynomial $f(x)$ is $y = -F_{[1,1]}^{-1}F_{[1]}$, because the linear part of

$$f(x+y) = \frac{1}{2}x^t F_{[1,1]}x + \left[F_{[1,1]}y + F_{[1]}\right]^t x + \left[\frac{1}{2}y^t F_{[1,1]}y + F_{[1]}^t y + F_{[0]}\right]$$

is zero. The translation part of the matching Euclidean transformation is obtained as the difference between the two centers, i.e. the best matching coordinate transformation should make the two centers coincide. The coordinate axes of the intrinsic frame of reference of a nonsingular quadric surface are the eigenvectors of the associated symmetric matrix. The best matching rotation makes the eigenvectors corresponding to the same eigenvalues coincide. If the eigenvalues are not repeated there is essentially a single solution, but in fact there are four solutions due to the symmetries of the quadratic surfaces (eight solutions if we also consider reflections). This construction not only applies to 3D surfaces, but also to 2D curves and quadric hypersurfaces of higher dimension. As we mentioned above, the equality condition on the eigenvalues is just a necessary condition for congruence. A sufficient condition is the equality (except for a multiplicative factor) of the defining polynomials recomputed with respect to the corresponding intrinsic frames of reference. However, computing the invariants, i.e. the eigenvalues, is computationally much less expensive than computing the full intrinsic coordinate system and then recomputing the coefficients of the defining polynomials in this new coordinate system.

The main contributions of this paper are the extension of these results to algebraic curves and surfaces of higher degree. In the first place we develop efficient techniques for computing Euclidean invariants of algebraic curves and surfaces as eigenvalues of certain matrices constructed from the coefficients of the defining polynomials, and show how to extend these methods to the affine and projective cases as well. The second contribution is how to extend the computation of the intrinsic Euclidean frame of reference, center and orientation, from quadratic surfaces to algebraic curves and surfaces of higher degree. This intrinsic coordinate system is independent of the viewer coordinate system, in the sense that the polynomial equations of the same curve or surface in its intrinsic coordinate system are independent of the viewer coordinate system, i.e. they are Euclidean invariants. As in the case of quadric surfaces, it is a necessary condition for two surfaces of the same degree to be congruent to have the same invariants, but a sufficient condition is to have the same equations in their intrinsic coordinate systems. So, this is the ultimate test for matching. However, when trying to find a match for one surface within a large data base of surfaces of the same degree, we start by reducing the number of candidates by comparing the invariants. Then, if matching candidates are found by comparing the invariants, the intrinsic frame of reference is computed and the coefficients evaluated in the new coordinate system. After this step, all the coefficients of the defining polynomials are compared with the candidates, and only those with similar coefficients are kept as valid hypotheses. Finally, we emphasize the *computational* aspect of these processes, which are based on both symbolic computations and well-known efficient, and numerically stable matrix algorithms.

With the methods introduced in this paper, a system that involves indexing into a data base of objects represented by features consisting of groups of moderately high degree algebraic surfaces can be developed, and the performance of many related recognition algorithms (Bolles *et al.*, 1983; Faugeras *et al.*, 1983; Grimson and Lozano-Perez, 1984; Bolles and Horaud, 1986; Faugeras and Hebert, 1986; Grimson and Lozano-Perez, 1987; Kishon and Wolfson, 1987; Schwartz and Sharir, 1987; Hong and Wolfson, 1988; Lamdan *et al.*, 1988; Lamdan and Wolfson, 1988; Bolle *et al.*, 1989a, 1989b; Chen and Kak, 1989; Grimson, 1989; Wolfson, 1990) can be improved.

The paper is organized as follows. In section 2 we review some basic concepts about implicit curves and surfaces, in particular about algebraic curves and surfaces. In section 3 we briefly describe how regions can be chosen. In section 4 we describe our previous work on implicit curve and surface fitting. In section 5 we describe in more detail our approach to classification and positioning based on the computation of invariants and covariants, and describe in detail our methods for fast evaluation of invariants. In section 6 we give a short recount of the history of invariant theory. In section 7 we describe how to compute the intrinsic Euclidean center of an algebraic 2D curve or 3D surface, while in section 8 we do the same for the intrinsic Euclidean orientation. In section 9 we extend these results to 3D curves. In section 10 we relate some of these invariants to previous results on algebraic curve and surface fitting. In section 11 we present some experimental results. Finally, in the appendix we include the longer proofs of the lemmas from the body of the paper.

## 2. Implicit curves and surfaces

An implicit surface is the set of zeros of a smooth function $f : \mathcal{R}^3 \to \mathcal{R}$ of three variables

$$Z(f) = \{(x_1, x_2, x_3) : f(x_1, x_2, x_3) = 0\} .$$

For example, figure 2 shows an implicit surface which is the set of zeros of the fourth degree polynomial $f(x_1, x_2, x_3) = x_1^4 - \frac{3}{4}x_1^2(x_2^2 + x_3^2) + (x_2^2 + x_3^2)^2 - 1$. Similarly, an implicit 2D curve is the set $Z(f) = \{(x_1, x_2) : f(x_1, x_2) = 0\}$ of zeros of a smooth function $f : \mathcal{R}^2 \to \mathcal{R}$ of two variables, and an implicit 3D curve is the intersection of two surfaces, the set $Z(\mathbf{f}) = \{(x_1, x_2, x_3) : \mathbf{f}(x_1, x_2, x_3) = 0\}$ of zeros of a two dimensional vector function $\mathbf{f} : \mathcal{R}^3 \to \mathcal{R}^2$ of three variables.

The representation of curves and surfaces in implicit form, as opposed to parametric form, has many advantages. In the first place, an implicit curve or surface maintains its

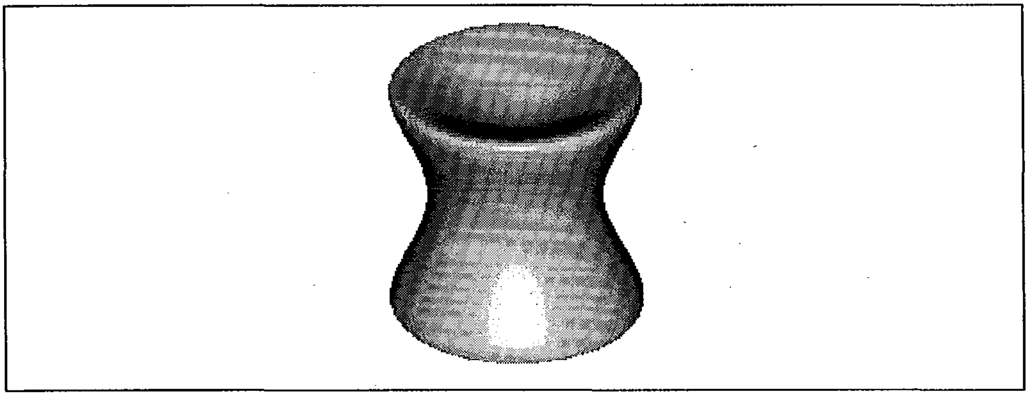

**Figure 2.** Implicit surface defined by the fourth degree polynomial $f(x_1, x_2, x_3) = x_1^4 - \frac{3}{4}x_1^2(x_2^2 + x_3^2) + (x_2^2 + x_3^2)^2 - 1$.

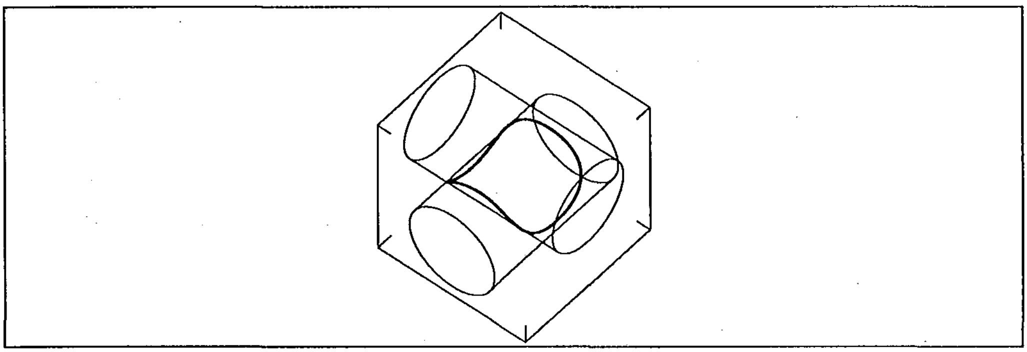

**Figure 3.** Two cylinders as a single fourth degree surface, and their nonplanar intersection curve.

implicit form after a change of coordinates, that is, if a set of points can be represented as a subset of an implicit curve or surface in one coordinate system, so can it be in any other coordinate system. That is not the case with data sets represented as graphs of functions of two variables, i.e. as depth maps, the patch descriptors produced by many well-known segmentation algorithms. In the second place, the union of two or more implicit curves or surfaces can be represented as a single implicit curve or surface, the set of zeros of the product of the functions which define the individual curves or surfaces

$$Z(f_1) \cup Z(f_2) \cup \cdots \cup Z(f_n) \ = \ Z(f_1 \cdot f_2 \cdots f_n) \,,$$

so that groups of curve or surface patches, or eventually a whole object, can be represented as a subset of a single implicit curve or surface. For example, the union of two cylinders

$$\{x : x_1^2 + (x_3 - 1)^2 - 4 = 0\} \cup \{x : x_2^2 + (x_3 + 1)^2 - 4 = 0\}$$

shown in figure 3, is the surface defined by the set of zeros of the product

$$\{x : (x_1^2 + (x_3 - 1)^2 - 4)(x_2^2 + (x_3 + 1)^2 - 4) = 0\} \,. \tag{2.1}$$

Hence, a single fourth degree polynomial can represent a pair of cylinders, and this is true for arbitrary cylinders, e.g. a pair that do not intersect. This property relaxes the

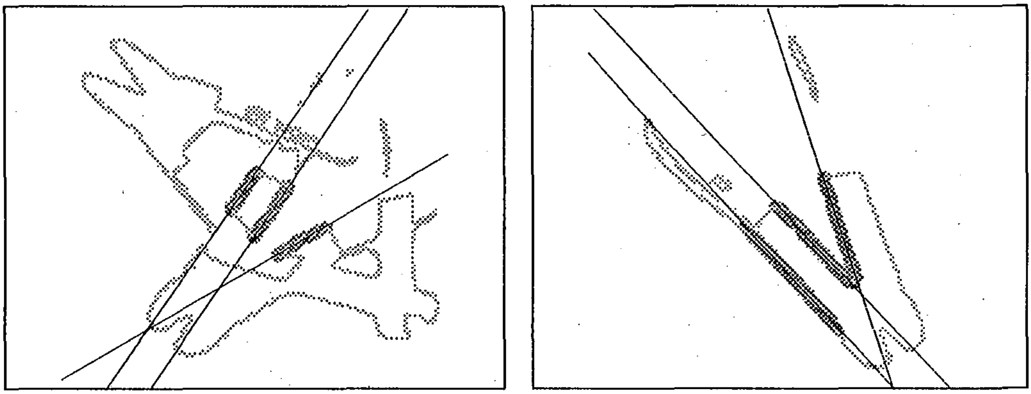

**Figure 4.** Two matching cubic 2D curves, unions of the three straight lines, fitted to the data points in the dark regions.

requirements on a segmentation algorithm, and it is very important in regard to the matching problem, allowing the matching of groups of patches at once.

## 3. Choosing regions

The high degree algebraic surfaces have much more discriminating power than does an individual low degree algebraic surface such as a plane or a quadric surface. Two types of representations are presently under consideration. One is the representation of a collection of a few low degree algebraic surfaces by a single algebraic surface of higher degree. For example, representing three planes by the set of zeros of a single third degree polynomial, the product of the three first degree polynomials, each representing one plane, or a quadric and a cubic surface by a single fifth degree algebraic surface. The simple low degree primitive surfaces used are those that can be found with modest computation. Exact segmentation is not necessary. Partial occlusion is not a problem; a primitive surface can be estimated from a portion of the primitive surface data. Once the primitives are found in the data, groups are then represented by single higher degree algebraic surfaces. Figure 4 shows two examples of this method. The other type of representation are the interest regions. These are spherical regions in which the data is not well represented by a low degree algebraic surface, such as first or second degree, but is well approximated by an algebraic surface of one degree higher. For example, a region occupied by a portion of two intersecting cylinders would be represented exactly by a fourth degree surface and poorly by a lower degree surface if enough of the surfaces were sensed. More generally, a fourth degree surface might capture a chunk of information useful for recognition purposes on a natural irregular surface such as a face, whereas a lower degree surface might not. Useful interest regions are those having the stability that the polynomial does not depend on the exact placement of the sphere specifying the region of data to be used. For this approach, sphere sizes should be chosen such that most spheres will contain data well approximated by low degree surfaces, and only a few will require representation by higher degree surfaces. These higher degree surfaces then contain considerable discriminatory power for object recognition. Figure 5 shows an example of a planar interest region. In this way we can deal with the occlusion problem. Note that the members of a group of

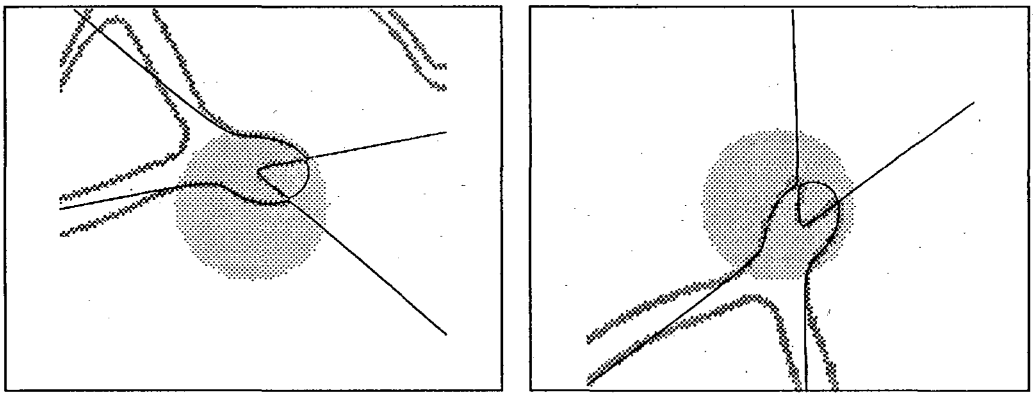

**Figure 5.** Interest regions, matching quartic 2D curves fitted to the data points inside the grey circles.

detected patches do not even have to be connected, so that hypotheses of objects and their positions can be generated from more global information, and this procedure can be implemented using a voting scheme, such as a generalized Hough transform or geometric hashing (Ballard, 1981; Lamdan and Wolfson, 1988; Bolle *et al.*, 1989a; Taubin, 1989).

## 4. Implicit curve and surface fitting

Toward building a recognition and positioning system based on implicit curves and surfaces, the first problem to deal with is how to fit implicit curves and surfaces to data. Several methods are well-known for extracting straight line segments (Duda and Hart, 1973), planar patches (Faugeras *et al.*, 1983), quadratic arcs (Paton, 1970a, 1970b; Biggerstaff, 1972; Albano, 1974; Turner, 1974; Cooper and Yalabik, 1975, 1976; Gnanadesikan, 1977; Bookstein, 1979; Sampson, 1982; Forsyth *et al.*, 1990) and quadric surface patches (Gennery, 1980; Hall *et al.*, 1982; Faugeras *et al.*, 1983; Cernuschi-Frias, 1984; Bolle and Cooper, 1984, 1986) from 2D edge maps and 3D range images. Some researchers have also developed methods for fitting algebraic curve and surface patches of arbitrary degree (Pratt, 1987; Chen, 1989; Taubin, 1988a, 1988b, 1990a). In this section we review some of Taubin's results.

The first step is to restrict the functions which define the curves or surfaces to a family parameterized by a finite number of parameters. Let $\phi : \mathcal{R}^{r+n} \to \mathcal{R}^k$ be a smooth function, and let us consider the maps $\mathbf{f} : \mathcal{R}^n \to \mathcal{R}^k$ which can be written as

$$\mathbf{f}(x) \equiv \phi(u, x) ,$$

for certain $u = (u_1, \ldots, u_r)^t$, in which case we will also write $\mathbf{f} = \phi_u$. We will refer to $u_1, \ldots, u_r$ as the *parameters* and to $x_1, \ldots, x_n$ as the *variables*. The family of all such maps will be denoted

$$\mathcal{F} = \{\mathbf{f} : \exists u \, \mathbf{f} = \phi_u\} ,$$

and we will say that $\phi$ is the *parameterization* of the family $\mathcal{F}$. The set of zeros $Z(\mathbf{f})$ of a member $\mathbf{f}$ of $\mathcal{F}$ is a 2D curve when $n = 2$ and $k = 1$, it is a surface when $n = 3$ and $k = 1$, and it is a 3D curve when $n = 3$ and $k = 2$.

Given a finite set of $n$-dimensional ( $n = 2$ or $n = 3$ ) data points $\mathcal{D} = \{p_1, \ldots, p_q\}$ , we would ideally fit an implicit curve or surface $Z(\mathbf{f})$ to the data set $\mathcal{D}$ by computing the minimizer $\hat{\mathbf{f}} \in \mathcal{F}$ of the mean square distance

$$\frac{1}{q} \sum_{i=1}^{q} \mathrm{dist}(p_i, Z(\mathbf{f}))^2 \tag{4.1}$$

from the data points to the curve or surface $Z(\mathbf{f})$ .

Unfortunately, there is no closed form expression for the distance from a point to a generic implicit curve or surface, not even for algebraic curves or surfaces, and iterative methods are required to compute it. This makes the minimization of (4.1) computationally impractical. In order to solve this problem we replace the real distance from a point to an implicit curve or surface by a first order approximation. The mean value of this function on a fixed set of data points is a smooth nonlinear function of the parameters, and can be *locally* minimized using well established non-linear least squares techniques.

The distance from a point $x \in \mathcal{R}^n$ to the set of zeros $Z(\mathbf{f})$ can be computed by direct methods only if the function $\mathbf{f}$ is linear, i.e. a first degree polynomial. In this case the Jacobian matrix $\mathrm{D}\mathbf{f}(x)$ is constant, and the following identity is satisfied

$$\mathbf{f}(y) \equiv \mathbf{f}(x) + \mathrm{D}\mathbf{f}(x) \cdot (y - x) .$$

Note that for a 3D surface or 2D curve, $\mathrm{D}\mathbf{f}(x) = \nabla f(x)^t$ , and for a 3D curve, $\mathrm{D}\mathbf{f}(x)$ is a two row matrix where each row is a transposed gradient vector. The unique point $\hat{y} \in Z(\mathbf{f})$ which minimizes the distance $\|y - x\|$ to $x$ , is given by

$$\hat{y} = x - [\mathrm{D}\mathbf{f}(x)]^{\dagger}\mathbf{f}(x) ,$$

where $[\mathrm{D}\mathbf{f}(x)]^{\dagger}$ is the *pseudoinverse* (Duda and Hart, 1973; Golub, 1983) of $\mathrm{D}\mathbf{f}(x)$ , so that the square of the distance from $x$ to $Z(\mathbf{f})$ is

$$\mathrm{dist}(x, Z(\mathbf{f}))^2 = \mathbf{f}(x)^t[\mathrm{D}\mathbf{f}(x)\,\mathrm{D}\mathbf{f}(x)^t]^{-1}\mathbf{f}(x) .$$

In the general case, where $\mathbf{f}(x)$ is not a first degree polynomial, we do not have an identity, but an approximation

$$\mathrm{dist}(x, Z(\mathbf{f}))^2 \approx \mathbf{f}(x)^t[\mathrm{D}\mathbf{f}(x)\,\mathrm{D}\mathbf{f}(x)^t]^{-1}\mathbf{f}(x) . \tag{4.2}$$

For $k = 1$ , the case of a 2D curve or 3D surface, the Jacobian has only one row, and (4.2) reduces to

$$\mathrm{dist}(x, Z(f))^2 \approx \frac{f(x)^2}{\|\nabla f(x)\|^2} . \tag{4.3}$$

Note that this approximate distance is the value of the function scaled down by the rate of growth at the point. Due to lack of space, we will continue the development for 2D curves and 3D surfaces ( $k = 1$ ), but all the results extend to 3D curves as well.

Now, we will fit curves or surfaces to data points by minimizing the *approximate mean square distance* from the data set $\mathcal{D}$ to the set of zeros of $f = \phi_u \in \mathcal{F}$

$$\Delta^2_{\mathcal{D}}(u) = \frac{1}{q} \sum_{i=1}^{q} \frac{f(p_i)^2}{\|\nabla f(p_i)\|^2} . \tag{4.4}$$

This expression can be seen as a sum of squares of smooth functions of the parameters.

The *local minimization* of (4.4) is a nonlinear least squares problem, which can be solved using iterative methods (Dennis and Shnabel, 1973) such as the Levenberg-Marquardt algorithm (Levenberg, 1944; Marquardt, 1963; Moré *et al.*, 1980).

In related work, Bajcsy and Solina (1987), Solina (1987), and Gross and Boult (1988) fit superquadrics (Barr, 1981) to data using the Levenberg-Marquardt algorithm. Also, Ponce and Kriegman (1989) use the Levenberg-Marquardt algorithm for fitting the projections of the occluding boundaries of algebraic surfaces to 2D edges.

Since we are interested in the *global minimization* of (4.4), and we want to avoid a global search, we present a method to choose a good initial estimate. We will only consider the linear model here, which corresponds to the case of algebraic curves or surfaces. In the linear model the maps can be written as

$$f(x) \;=\; F_1 X_1(x) + \cdots + F_h X_h(x) \;=\; F X(x) \;,$$

where $F = (F_1, \ldots, F_h)$ is a row vector of coefficients, $X = (X_1, \ldots, X_h)^t : \mathcal{R}^n \to \mathcal{R}^h$ is a fixed map, and the parameter vector is just $u = F^t$. In order to find a good initial estimate for the linear model, we replace the performance function. Instead of the approximate mean square distance (4.4) we use a new approximation, turning the difficult multimodal optimization problem into a generalized eigenproblem.

There exist certain families of implicit curves or surfaces, such as those which define straight lines, circles, planes, spheres and cylinders, which have the value of $\|\nabla f(x)\|^2$ constant on $Z(f)$. In those cases we have

$$\frac{1}{q}\sum_{i=1}^{q} \frac{f(p_i)^2}{\|\nabla f(p_i)\|^2} \approx \frac{\frac{1}{q}\sum_{i=1}^{q} f(p_i)^2}{\frac{1}{q}\sum_{i=1}^{q} \|\nabla f(p_i)\|^2} \,,$$

In the linear model, the right hand side of the previous expression reduces to the quotient of two quadratic functions of the parameters

$$\frac{\frac{1}{q}\sum_{i=1}^{q} f(p_i)^2}{\frac{1}{q}\sum_{i=1}^{q} \|\nabla f(p_i)\|^2} = \frac{FMF^t}{FNF^t} \,, \tag{4.5}$$

where the matrices $M$ and $N$ are non-negative definite, symmetric, and only functions of the data points:

$$M \;=\; \frac{1}{q}\sum_{i=1}^{q}[X(p_i)X(p_i)^t] \qquad N \;=\; \frac{1}{q}\sum_{i=1}^{q}[\mathrm{D}X(p_i)\mathrm{D}X(p_i)^t] \,.$$

The new problem, the minimization of (4.5), reduces to a generalized eigenvalue problem, with the minimizer being the eigenvector corresponding to the minimum eigenvalue of the pencil $F(M - \lambda N) \;=\; 0$.

This *generalized eigenvector fit* can be extended to 3D curves as well, where the solution is given by the eigenvectors corresponding to the *two* least eigenvalues. For example, figure 6 shows the result of fitting an implicit 3D curve, defined by the intersection of two general quadric surfaces, to the data points using the generalized eigenvector fit algorithm.

For fitting at modest computational cost, our approach is to use the *generalized eigenvector fit* method first, also in the general case where $\|\nabla f(x)\|^2$ is not constant on $Z(f)$,

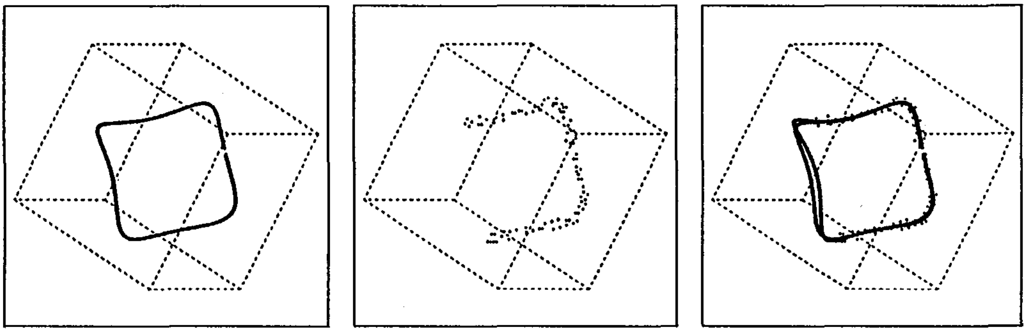


**Figure 6.** 3D curve fitted to data using the generalized eigenvector fit algorithm. (a) Original curve. (b) Data generated close to the original curve. (c) Superposition of original curve, generalized eigenvector fit, and data.

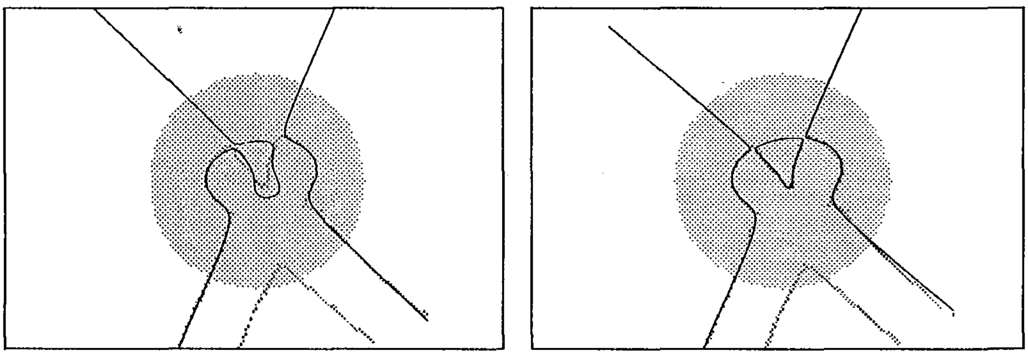

**Figure 7.** (a) Fourth degree algebraic curve fitted to edge data inside the circular region using the generalized eigenvector fit algorithm. (b) Improvement after using the previous result as initial value in the minimization of the approximate mean square distance.

and subject the result to a statistical test. If the test is satisfied, the fitting process stops here, otherwise the result of the generalized eigenvector fit is used as a starting point in the local minimization of (4.4). Finally, it is important to mention that the curves or surfaces computed with the *generalized eigenvector fit* method are often satisfactory, not requiring further improvement, and the required computation is modest and practical.

## 5. Invariants of algebraic curves and surfaces

If $x' = T(x)$ is a nonsingular coordinate transformation, Euclidean, affine, or projective depending on the application, and $\mathbf{f}(x)$ is a polynomial, we will denote by $\mathbf{f}'(x')$ the unique polynomial which satisfies the polynomial identity $\mathbf{f}(x) \equiv \mathbf{f}'(x'))$, i.e. $\mathbf{f}(x) \equiv \mathbf{f}'(T(x))$, or equivalently, $\mathbf{f}'(x') \equiv \mathbf{f}(T^{-1}(x'))$ . In this way we can look at both $Z(\mathbf{f}') = \{x' : \mathbf{f}'(x') = 0\}$ and $Z(\mathbf{f}) = \{x : \mathbf{f}(x) = 0\}$ as describing the same set of points, but in the two different coordinate systems. The coefficients of $\mathbf{f}'(x')$ can be computed as functions of the coefficients of $\mathbf{f}(x)$ and the parameters of the transformation $T(x)$, but the applications to object recognition need to solve the inverse problem: the transformation parameters must be recovered from the two polynomials. Two polynomials $\mathbf{f}$ and $\mathbf{f}'$ of the same degree are called *congruent* if there exists a coordinate transformation $x' = T(x)$ such that $\mathbf{f}(x) \equiv \mathbf{f}'(T(x))$. Given two polynomials of the same degree, we have to decide, by just looking at the coefficients of both polynomials, whether they are congruent or not. In fact, in order to significantly reduce the computation involved in the data base search, we only need to be able to check at a low computational cost certain necessary conditions for congruence. These necessary conditions for congruence have to be such that only a small subset of the data base will satisfy them. Then, we can check at a higher computational cost sufficient conditions for congruence on the small subset of the data base which satisfies the first set of conditions. Also, due to the finiteness of the database and the numerical and measurement errors involved, we seek approximate answers. The first step in the classification problem, checking necessary conditions for congruence, can be solved using invariants. Broadly speaking, an invariant is a function $\mathcal{I}(\mathbf{f})$ of the coefficients of $\mathbf{f}$ which does not change when a coordinate transformation $x' = T(x)$ is applied, i.e. $\mathcal{I}(\mathbf{f}) = \mathcal{I}(\mathbf{f}')$. In our object recognition application, a sufficiently long vector $\mathcal{I}(\mathbf{f}) = (\mathcal{I}_1(\mathbf{f}), \ldots \mathcal{I}_s(\mathbf{f}))^t$ of invariants will be used. Those elements of the data base whose invariant vectors are *close* to the invariant vector corresponding to the measured polynomial will be considered as candidates to undergo the more expensive check of sufficient conditions. Also, the invariants can be used to organize the data base for an efficient search. A rough quantization of invariant space into uniformly occupied cells can be used to define a hash function which reduces the number of comparisons needed to classify a given polynomial $\mathbf{f}$. In this section we introduce techniques for *computing* invariants of polynomials at low computational cost. Our methods are based on reducing the problem of computing invariants of polynomials to the computation of eigenvalues of certain associated matrices, because very efficient and well understood numerical methods are available for this operation (Smith *et al.*, 1976; Garbow *et al.*, 1977; Golub and Van Loan, 1983).

## 5.1. Polynomials and Forms

From now on, polynomials will be written expanded in Taylor series at the origin

$$f(x) = \sum_{\alpha} \frac{1}{\alpha!} F_\alpha x^\alpha , \tag{5.1}$$

where the vector of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_n)^t$ is a *multiindex* of *size* $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$ is a multiindex factorial, $F_\alpha$ is a coefficient of degree $|\alpha|$, and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is a *monomial* of degree $|\alpha|$. There are exactly $h_d = \binom{n+d-1}{n-1} = \binom{n+d-1}{d}$ different multiindices of size $d$, and so, that many monomials of degree $d$. A polynomial of degree $d$ in $n$ variables has $h_d + h_{d-1} + \cdots + h_0 = \binom{n+d}{n}$ coefficients, as many as a form of the same degree but in $n+1$ variables. The coefficients of $f$ are equal to the partial derivatives of order $d$ evaluated at the origin

$$F_\alpha = \left. \frac{\partial^{\alpha_1 + \cdots + \alpha_n} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right|_{x=0}$$

and only finitely many coefficients are different from zero. A polynomial is homogeneous, or a *form*, if all its terms are of the same degree

$$\phi(x) = \sum_{|\alpha|=d} \tfrac{1}{\alpha!}\, \Phi_\alpha\, x^\alpha \,. \tag{5.2}$$

For example, a fourth degree form in three variables is written as

$$\begin{array}{l} \phi(x_1,x_2,x_3) \;= \\ \begin{array}{llllll} \frac{1}{24}\,\Phi_{(4,0,0)}\,x_1^4 & + & \frac{1}{6}\,\Phi_{(3,1,0)}\,x_1^3x_2 & + & \frac{1}{6}\,\Phi_{(3,0,1)}\,x_1^3x_3 & + \\ \frac{1}{4}\,\Phi_{(2,2,0)}\,x_1^2x_2^2 & + & \frac{1}{2}\,\Phi_{(2,1,1)}\,x_1^2x_2x_3 & + & \frac{1}{4}\,\Phi_{(2,0,2)}\,x_1^2x_3^2 & + \\ \frac{1}{6}\,\Phi_{(1,3,0)}\,x_1x_2^3 & + & \frac{1}{2}\,\Phi_{(1,2,1)}\,x_1x_2^2x_3 & + & \frac{1}{2}\,\Phi_{(1,1,2)}\,x_1x_2x_3^2 & + \\ \frac{1}{6}\,\Phi_{(1,0,3)}\,x_1x_3^3 & + & \frac{1}{24}\,\Phi_{(0,4,0)}\,x_2^4 & + & \frac{1}{6}\,\Phi_{(0,3,1)}\,x_2^3x_3 & + \\ \frac{1}{4}\,\Phi_{(0,2,2)}\,x_2^2x_3^2 & + & \frac{1}{6}\,\Phi_{(0,1,3)}\,x_2x_3^3 & + & \frac{1}{24}\,\Phi_{(0,0,4)}\,x_3^4\,. & \end{array} \end{array} \tag{5.3}$$

### 5.2. Invariants of polynomials and forms

A function $\mathcal{I}(\phi)$ of the coefficients of a form $\phi$ of degree $d$ is a *relative invariant* of weight $w$ if for every nonsingular coordinate transformation $x' = Ax$, we have $\mathcal{I}(\phi') = |A|^{-w}\mathcal{I}(\phi)$, where $\phi'(x') = \phi(A^{-1}x')$. An *absolute invariant* is an invariant of weight zero. If $\mathcal{I}(\phi)$ is a polynomial function of the coefficients of $\phi$ we talk about a *polynomial invariant*, and of a *rational invariant* if the function $\mathcal{I}$ is the ratio of two polynomial functions. Join invariants of several forms can be defined in a similar way, and in the Euclidean case, all the invariants are absolute.

The techniques and algorithms presented below provide methods to compute invariants of a form, or join invariants of several forms, with respect to either orthogonal or homogeneous linear transformations, i.e. coordinate transformations that can be written as $x' = Ax$, with the matrix $A$ being either orthogonal, or just nonsingular. Although projective coordinate transformations are defined by linear transformations, affine and Euclidean coordinate transformations are non-homogeneous. In this section we discuss methods to reduce the problem of computing affine and Euclidean invariants of polynomials to the computation of orthogonal and linear invariants of certain forms. There are basically three approaches to this. The first one is to introduce homogeneous coordinates, and look at the groups of affine and Euclidean transformations as subgroups of the projective group. Every invariant with respect to the projective group is also invariant with respect to the affine and Euclidean subgroups. However, these are not all. The second approach is based on the observation that when an affine coordinate transformation is applied, the terms of higher degree of a polynomial, the leading form, are independent of the translation part. The third approach is to use a covariant vector, a vector whose coordinates are known in every coordinate system, as the coordinate system origin, to reduce the problem of computing affine or Euclidean invariants of a polynomial to the computation of join linear or orthogonal invariants of a finite number of forms.

By introducing homogeneous coordinates, every curve or surface described in Euclidean space by a polynomial in $n$ variables, can be described in projective space by an associated homogeneous polynomial in $n+1$ variables. If $\phi(x_0,\ldots,x_n)$ is a form of degree $d$ in $n+1$ variables, and $f(v_1,\ldots,v_n)$ is a regular polynomial of degree $\le d$ in $n$

variables, the one-to-one correspondence is given by

$$\begin{aligned} \phi(x_0,\ldots,x_n) &= x_0^d f(x_1/x_0,\ldots,x_n/x_0) \\ f(v_1,\ldots,v_n) &= \phi(1,v_1,\ldots,v_n) . \end{aligned}$$

In other words, every polynomial in $n$ variables is the restriction of a form in $n+1$ variables to the hyperplane $\{x : x_0 = 1\}$, and every form in $n+1$ variables is totally determined by its restriction to this hyperplane. A projective transformation can be written as a homogeneous linear transformation $x' = Ax$ on the homogeneous coordinates of a point. Every nonsingular matrix $A$ defines a projective transformation, but the correspondence is not one-to-one. Two nonsingular matrices $A$ and $B$ define the same projective transformation if $A = \lambda B$ for certain constant $\lambda \neq 0$. If $v' = Av + b$ is an affine coordinate transformation, the corresponding projective transformation is given by

$$x' = \begin{pmatrix} 1 & 0 \\ b & A \end{pmatrix} x .$$

Given a point $y \in \mathcal{R}^n$, every polynomial $f$ of degree $d$ can be written as a sum of forms

$$f(x) = \sum_{k=0}^{d} f_k(x+y) ,$$

where $f_k(x)$ is a form of degree $k$, and $f_d$ is not identically zero. This representation is unique, provided that $y$ is kept fixed. Using a counting argument on the degrees of the terms, it is not difficult to see that the form of degree $d$, the *leading form*, is independent of $y$, so that every invariant of $f_d$ with respect to linear or orthogonal transformations $x' = Ax$ is an invariant of $f$ with respect to affine or Euclidean transformations.

Also, if $x' = Ax + b$ is an affine or Euclidean coordinate transformation, and $y' = Ay + b$, we can rewrite the coordinate transformation as $x' - y' = A(x - y)$. That is, if we know beforehand the coordinates of a point in both coordinates systems, we can just consider homogeneous linear transformations, by restricting the coordinate systems to those with the origin at the known point. In this way the problem of computing affine or Euclidean invariants of the polynomial $f$ can be reduced to computing join invariants of the forms $f_0,\ldots,f_d$ with respect to linear or orthogonal transformations $x' = Ax$. We just need to find a vector function $y_f = y(f)$ of the coefficients of $f$ such that if $x' = Ax + b$ is an affine or Euclidean transformation, and $f'(x') = f(A^{-1}(x' - b))$ is the polynomial which describes the zeros of $f$ in the new coordinate system, then $y_{f'} = A(y_f + b)$. Such a function is a particular case of what is classically called a *covariant* vector of weight zero. We will show how to compute such a covariant vector in the Euclidean case.

In the affine case, an alternative is to compute the vectors $y$ and $y'$ from the data sets used to estimate the coefficients of the polynomials, and estimate the coefficients with the origin at that point. In this way we do not have to recompute the coefficients with respect to the center. For example, the centroids are good candidates. Other features such as high curvature points, or intersections points of lower degree curves or surfaces, can also be used, depending on the application.

### 5.3. Coefficients and linear transformations

Let $\phi(x)$ be a quadratic form in three variables

$$\phi\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \frac{1}{2}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}^t \begin{pmatrix}\Phi_{(2,0,0)} & \Phi_{(1,1,0)} & \Phi_{(1,0,1)}\\ \Phi_{(1,1,0)} & \Phi_{(0,2,0)} & \Phi_{(0,1,1)}\\ \Phi_{(1,0,1)} & \Phi_{(0,1,1)} & \Phi_{(0,0,2)}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x3\end{pmatrix}.$$

If we denote the 3×3 square matrix $\Phi_{[1,1]}$, we can write it just as $\phi(x) = \frac{1}{2}x^t\Phi_{[1,1]}x$. If $x' = Ax$ is a linear transformation, and $\phi'(x') = \phi(A^{-1}x') = \frac{1}{2}x'^t\Phi'_{[1,1]}x'$, the two matrices of coefficients are related by the formula $\Phi'_{[1,1]} = A^{-t}\,\Phi_{[1,1]}\,A^{-1}$, which reduces to $\Phi'_{[1,1]} = A\,\Phi_{[1,1]}\,A^t$ if the matrix $A$ is orthogonal. In general, for $A$ nonsingular we have $|\Phi_{[1,1]}| = |A|^{-2}|\Phi'_{[1,1]}|$, and for $A$ orthogonal it is well-known that the coefficients of the characteristic polynomial $\chi(\lambda) = \det(\lambda I - \Phi_{[1,1]})$ are orthogonal invariants of the form $\phi$. Equivalently, the three eigenvalues of the matrix, the roots of $\chi(\lambda)$ are orthogonal invariants. Let $\phi(x) = \frac{1}{2}x^t\Phi_{[1,1]}x$ and $\psi(x) = \frac{1}{2}x^t\Psi_{[1,1]}x$ be two nonsingular quadratic forms in three variables. They describe planar conics in homogeneous coordinates. It is also well- known that $\mathrm{trace}(\Phi_{[1,1]}\Psi^{-1}_{[1,1]})$ and $\det(\Phi_{[1,1]}\Psi^{-1}_{[1,1]})$ are join rational (ratio of two polynomials) projective invariants of the two forms. In fact, these invariants are just two of the coefficients of the characteristic polynomial $\chi(\lambda) = \det(\lambda I - \Phi_{[1,1]}\Psi^{-1}_{[1,1]})$, and all the coefficients of $\chi(\lambda)$ are join linear invariants of the two forms. Equivalently, the three eigenvalues of the matrix $\Phi_{[1,1]}\Psi^{-1}_{[1,1]}$ are join invariants of the two forms. Finally, this construction can be extended to all the pairs of quadratic forms, not only the nonsingular ones; instead of considering the eigenvalues of $\Phi_{[1,1]}\Psi^{-1}_{[1,1]}$, consider the generalized eigenvalues of the pair of matrices, the values of $(\lambda,\nu) \neq 0$ such that $\det(\lambda\Phi_{[1,1]} + \nu\Psi_{[1,1]}) = 0$.

We will show how to generalize the preceding constructions to higher degrees. That is, we will show how to compute orthogonal and linear invariants of one or more forms by reducing the problem to the computation of eigenvalues or generalized eigenvalues of certain matrices of coefficients. But in order to do so, we first have to introduce the proper terminology and establish certain basic properties.

Multiindices can be linearly ordered in many different ways. We will only use the (inverse) *lexicographical order*, but the same results can be obtained using other orders. If $\alpha$ and $\beta$ are two multiindices of the same size, we say that $\alpha$ precedes $\beta$, and write $\alpha < \beta$ if, for the first index $k$ such that $\alpha_k$ differs from $\beta_k$, we have $\alpha_k > \beta_k$. For example, for multiindices of size 2 in three variables, the lexicographical order is

$$(2,0,0) < (1,1,0) < (1,0,1) < (0,2,0) < (0,1,1) < (0,0,2).$$

If $\alpha$ and $\beta$ are multiindices of different sizes, and the size of $\alpha$ is less than the size of $\beta$, we also say that $\alpha$ precedes $\beta$, and write $\alpha < \beta$.

The set of monomials $\{x^\alpha/\sqrt{\alpha!} : |\alpha| = d\}$ of degree $d$ lexicographically ordered, define a vector of dimension $h_d$, which we will denote $X_{[d]}(x)$. For example,

$$X_{[3]}(x_1,x_2) = \left(\tfrac{1}{\sqrt{6}}x_1^3 \quad \tfrac{1}{\sqrt{2}}x_1^2x_2 \quad \tfrac{1}{\sqrt{2}}x_1x_2^2 \quad \tfrac{1}{\sqrt{6}}x_2^3\right)^t.$$

For every pair of nonnegative integers $(j,k)$, we will denote $X_{[j,k]}(x)$ the $h_j \times h_k$ matrix $X_{[j]}(x)X_{[k]}^t(x)$. That is, $X_{[j,k]}(x)$ is the matrix defined by the set of monomials $\{x^{\alpha+\beta}/\sqrt{\alpha!\,\beta!}\ :\ |\alpha| = j\ ,\ |\beta| = k\}$ of degree $d = j + k$, lexicographically ordered according to two multiindices. For example,

$$X_{[2,2]}(x_1, x_2, x_3) \;=$$

$$\begin{pmatrix}
\frac{1}{2}\,x_1^4 & \frac{1}{\sqrt{2}}\,x_1^3x_2 & \frac{1}{\sqrt{2}}\,x_1^3x_3 & \frac{1}{2}\,x_1^2x_2^2 & \frac{1}{\sqrt{2}}\,x_1^2x_2x_3 & \frac{1}{2}\,x_1^2x_3^2 \\
\frac{1}{\sqrt{2}}\,x_1^3x_2 & x_1^2x_2^2 & x_1^2x_2x_3 & \frac{1}{\sqrt{2}}\,x_1x_2^3 & x_1x_2^2x_3 & \frac{1}{\sqrt{2}}\,x_1x_2x_3^2 \\
\frac{1}{\sqrt{2}}\,x_1^3x_3 & x_1^2x_2x_3 & x_1^2x_3^2 & \frac{1}{\sqrt{2}}\,x_1x_2^2x_3 & x_1x_2x_3^2 & \frac{1}{\sqrt{2}}\,x_1x_3^3 \\
\frac{1}{2}\,x_1^2x_2^2 & \frac{1}{\sqrt{2}}\,x_1x_2^3 & \frac{1}{\sqrt{2}}\,x_1x_2^2x_3 & \frac{1}{2}\,x_2^4 & \frac{1}{\sqrt{2}}\,x_2^3x_3 & \frac{1}{2}\,x_2^2x_3^2 \\
\frac{1}{\sqrt{2}}\,x_1^2x_2x_3 & x_1x_2^2x_3 & x_1x_2x_3^2 & \frac{1}{\sqrt{2}}\,x_2^3x_3 & x_2^2x_3^2 & \frac{1}{\sqrt{2}}\,x_2x_3^3 \\
\frac{1}{2}\,x_1^2x_3^2 & \frac{1}{\sqrt{2}}\,x_1x_2x_3^2 & \frac{1}{\sqrt{2}}\,x_1x_3^3 & \frac{1}{2}\,x_2^2x_3^2 & \frac{1}{\sqrt{2}}\,x_2x_3^3 & \frac{1}{2}\,x_3^4
\end{pmatrix}$$

Consistently with this notation, the vector $\{\Phi_\alpha/\sqrt{\alpha!} : |\alpha| = d\}$ of coefficients of $\phi$ will be denoted $\Phi_{[d]}$. In this way, a form $\phi$ of degree $d$ can be written in vector form

$$\phi(x) \;=\; \Phi_{[d]}^t X_{[d]}(x)\ . \tag{5.4}$$

Also, for every pair of integers, $(j, k)$ such that $j + k = d$, the set of coefficients

$$\left\{ \sqrt{\tfrac{1}{\alpha!\,\beta!}}\,\Phi_{\alpha+\beta}\ :\ |\alpha| = j, |\beta| = k \right\}$$

lexicographically ordered in both indices, defines an $h_j \times h_k$ matrix which we will denote $\Phi_{[j,k]}(x)$. For example, for the fourth degree form in three variables (5.3) we have

$$\Phi_{[2,2]} \;=\; \begin{pmatrix}
\frac{1}{2}\Phi_{(4,0,0)} & \frac{1}{\sqrt{2}}\Phi_{(3,1,0)} & \frac{1}{\sqrt{2}}\Phi_{(3,0,1)} & \frac{1}{2}\Phi_{(2,2,0)} & \frac{1}{\sqrt{2}}\Phi_{(2,1,1)} & \frac{1}{2}\Phi_{(2,0,2)} \\
\frac{1}{\sqrt{2}}\Phi_{(3,1,0)} & \Phi_{(2,2,0)} & \Phi_{(2,1,1)} & \frac{1}{\sqrt{2}}\Phi_{(1,3,0)} & \Phi_{(1,2,1)} & \frac{1}{\sqrt{2}}\Phi_{(1,1,2)} \\
\frac{1}{\sqrt{2}}\Phi_{(3,0,1)} & \Phi_{(2,1,1)} & \Phi_{(2,0,2)} & \frac{1}{\sqrt{2}}\Phi_{(1,2,1)} & \Phi_{(1,1,2)} & \frac{1}{\sqrt{2}}\Phi_{(1,0,3)} \\
\frac{1}{2}\Phi_{(2,2,0)} & \frac{1}{\sqrt{2}}\Phi_{(1,3,0)} & \frac{1}{\sqrt{2}}\Phi_{(1,2,1)} & \frac{1}{2}\Phi_{(0,4,0)} & \frac{1}{\sqrt{2}}\Phi_{(0,3,1)} & \frac{1}{2}\Phi_{(0,2,2)} \\
\frac{1}{\sqrt{2}}\Phi_{(2,1,1)} & \Phi_{(1,2,1)} & \Phi_{(1,1,2)} & \frac{1}{\sqrt{2}}\Phi_{(0,3,1)} & \Phi_{(0,2,2)} & \frac{1}{\sqrt{2}}\Phi_{(0,1,3)} \\
\frac{1}{2}\Phi_{(2,0,2)} & \frac{1}{\sqrt{2}}\Phi_{(1,1,2)} & \frac{1}{\sqrt{2}}\Phi_{(1,0,3)} & \frac{1}{2}\Phi_{(0,2,2)} & \frac{1}{\sqrt{2}}\Phi_{(0,1,3)} & \frac{1}{2}\Phi_{(0,0,4)}
\end{pmatrix}$$

These matrices of coefficients let us rewrite forms as quadratics in the monomials of lower degrees, generalizing what we usually do for quadratic forms. Although the result will not be used in this paper, for completeness we enunciate Euler's theorem, but we omit the proof. For $j = 1$ the proof can be found in Walker (1950), and general case in Taubin (1991).

LEMMA 5.1. (EULER'S THEOREM) *For every form $\phi$ of degree $d = j+k$, we have*

$$\tbinom{d}{j}\phi(x) \;=\; X_{[j]}(x)^t\,\Phi_{[j,k]}X_{[k]}(x)\ .$$

If $x' = Ax$ is a nonsingular linear transformation, for every form $\psi(x)$, the polynomial $\psi(Ax)$ is a form of the same degree. In particular, every component of the vector $X_{[d]}(Ax)$ can be written in a *unique way* as a linear combination of the elements of $X_{[d]}(x)$, or in matrix form

$$X_{[d]}(Ax) \;=\; A_{[d]}X_{[d]}(x)\ ,$$

where $A_{[d]}$ is a nonsingular $h_d \times h_d$ matrix. We will call the map $A \mapsto A_{[d]}$ the *d-th degree representation*, and the matrix $A_{[d]}$ the *k-th degree representation matrix* of $A$. Furthermore,

LEMMA 5.2. *The map $A \mapsto A_{[d]}$ satisfies the following properties :*

1. *It defines a* faithful linear representation *(a $1-1$ homomorphism of groups) of the group of nonsingular $n \times n$ matrices $GL(n)$ into the group of nonsingular $h_d \times h_d$ matrices $GL(h_d)$. That is, for every pair of nonsingular matrices $A$, $B$, we have (a): $(AB)_{[d]} = A_{[d]}B_{[d]}$ (preserves products), (b): if $A_{[d]} = B_{[d]}$, then $A = B$ (is one to one), the matrix $A_{[d]}$ is nonsingular, and (c): $(A_{[d]})^{-1} = (A^{-1})_{[d]}$.*
2. *It preserves transposition, i.e. for every nonsingular matrix $A$, we have $(A^t)_{[d]} = (A_{[d]})^t$. In particular, if $A$ is symmetric, positive definite, or orthogonal, so is $A_{[d]}$.*
3. *If $A$ is lower triangular, so is $A_{[d]}$. In particular, if $A$ is diagonal, so is $A_{[d]}$.*
4. *The determinant of $A_{[d]}$ is equal to $|A|^m$, with $m = \binom{n+d-1}{n-1}$.*

PROOF. This is a well-known result in the theory of representations of Lie groups (Brockett, 1973), but for completeness, we include an elementary proof in the appendix. □

Now we can establish the relations between the vectors and matrices of coefficients corresponding to congruent forms. The representation matrices play a central role.

LEMMA 5.3. *If $\phi(x)$ is a form of degree $d$, $x' = Ax$ a nonsingular coordinate transformation, $\phi'(x') = \phi(A^{-1}x')$ is the form which describes the same curve or surface in the new coordinate system, and $j,k$ are two nonnegative integers such that $j+k=d$, then*

$$\Phi'_{[j,k]} \;=\; A_{[j]}^{-t}\Phi_{[j,k]}A_{[k]}^{-1}\,.$$

*In particular,* $\Phi'_{[d]} = A_{[d]}^{-t}\Phi_{[d]}$.

PROOF. In the appendix. □

In the particular case of orthogonal matrices, we have

COROLLARY 5.1. *With the same hypothesis of lemma 5.3. If the matrix $A$ is orthogonal*

$$\Phi'_{[j,k]} \;=\; A_{[j]}\Phi_{[j,k]}A_{[k]}^{t}\,.$$

*In particular,* $\Phi'_{[d]} = A_{[d]}\Phi_{[d]}$.

Now we have all the elements to start computing invariants of forms.

## 5.4. COMPUTATION OF LINEAR INVARIANTS

Our first method involves the computation of determinants. It is not very useful by itself, but it will provide the basis to derive more complex invariants.

LEMMA 5.4. *Let $\phi$ be a form of even degree $d=2k$. Then, the determinant $|\Phi_{[k,k]}|$ is a linear invariant of weight $w = 2\binom{n+d-1}{n}$. In particular, it is an absolute orthogonal invariant.*

PROOF. If $x' = Ax$ is a coordinate transformation, from lemma 5.3 we have $\Phi'_{[k,k]} = A_{[k]}^{-t}\Phi_{[k,k]}A_{[k]}^{-1}$, and taking determinants on both sides $|\Phi'_{[k,k]}| = |A_{[k]}|^{-2}|\Phi_{[k,k]}|$. But from lemma 5.2, $|A_{[k]}| = |A|^m$, with $m = \binom{n+d-1}{n}$. □

The second method is a consequence of the previous one. It applies to pairs of forms of the same degree under nonsingular linear transformations.

COROLLARY 5.2. *Let $\phi$ and $\psi$ be two forms of even degree $d = 2k$. Then, the coefficients of the homogeneous polynomial of two variables $\xi(\theta_1, \theta_2) = |\,\theta_1\Phi_{[k,k]} + \theta_2\Psi_{[k,k]}\,|$, are join rational invariants of the pair of forms. Equivalently, the generalized eigenvalues of the pair of square matrices are absolute invariants of the pair of forms. In particular, if the matrix $\Psi_{[k,k]}$ is nonsingular, then the eigenvalues of the matrix $\Phi_{[k,k]}\Psi_{[k,k]}^{-1}$ are absolute invariants of the pair.*

PROOF. Let $x' = Ax$ be a coordinate transformation, and let $\xi'(\theta_1, \theta_2) = |\,\theta_1\Phi'_{[k,k]} + \theta_2\Psi'_{[k,k]}\,|$. For each fixed value of $(\theta_1, \theta_2) \neq 0$, let us consider the new form $\theta_1\phi(x) + \theta_2\psi(x)$ of degree $d$. If we apply lemma 5.4 to this form, we obtain

$$\xi'(\theta_1, \theta_2) \;=\; |A|^{-2m}\xi(\theta_1, \theta_2)\,.$$

Since this is true for every value of $(\theta_1, \theta_2) \neq 0$, it is a polynomial identity, and so the coefficients of $|A|^{-2m}\xi$ and $\xi'$ corresponding to the same powers $\theta_1^{j_1}\theta_2^{j_2}$ coincide, i.e. the coefficients of $\xi$ are relative invariants of weight $2m$. The homogeneous polynomial $\xi$ of degree $h_k$ has exactly $h_k$ roots in the projective line. These are the generalized eigenvalues of the pair of matrices $\Phi_{[k,k]}$ and $\Psi_{[k,k]}$. Since $\xi' = |A|^{-2m}\xi$, they are clearly independent of coordinate transformations, and so, absolute invariants of the pair. □

The previous result can be extended, with basically the same proof, to join absolute invariants of many forms. We state the results without proofs.

COROLLARY 5.3. *Let $\phi_1, \ldots, \phi_r$ be forms of even degree $d = 2k$. Then, the coefficients of the homogeneous polynomial of $r$ variables $\xi(\theta_1, \ldots, \theta_r) = |\,\theta_1\Phi_{1[k,k]} + \cdots + \theta_r\Psi_{r[k,k]}\,|$, are join rational invariants of the forms.*

COROLLARY 5.4. *If $\phi_1, \psi_1 \ldots, \phi_r, \psi_r$ are forms of even degree $d = 2k$, and the matrices $\Psi_{1[k,k]}, \ldots, \Psi_{r[k,k]}$ are nonsingular, then the eigenvalues of the matrix*

$$\Phi_{1[k,k]}\Psi_{1[k,k]}^{-1} \cdots \Phi_{r[k,k]}\Psi_{r[k,k]}^{-1}$$

*are absolute join invariants.*

If $\phi$ and $\psi$ are not forms of even degree, we can apply the construction of corollary 5.2 to their squares $\phi^2$ and $\psi^2$. This is particularly interesting, even for forms of even degree, because the number of invariants obtained is equal to the number of coefficients. More generally, if $\phi$ is a form of degree $d_1$, and $\phi$ is a form of degree $d_2 \neq d_1$, we can apply corollary 5.2 to $\phi^{(d/d_1)}$ and $\psi^{(d/d_2)}$, where $d$ is equal to twice the minimum common multiple of $d_1$ and $d_2$. Clearly, this method can also be applied to the cases of many forms. However, in these cases we have the extra computational cost of evaluating the powers of the forms using symbolic methods. If the exponents are not too large, then this is also a practical way to obtain more invariants.

### 5.5. Covariant and contravariant matrices

Clearly the methods introduced in the previous section are only based on the properties of the representation matrices and the relations between matrices of coefficients of forms in different coordinate systems. In general a matrix $C_{[j,k]}$ whose components are functions of one or more forms $\phi, \psi, \ldots$, not necessarily of the same degrees, and such that

$$C_{[j,k]}(\phi', \psi', \ldots) = A_{[j]}^{-t}\, C_{[j,k]}(\phi, \psi, \ldots)\, A_{[k]}^{-1}$$

will be called a *covariant matrix*, and will be briefly denoted $C_{[j,k]}$, while the same matrix function evaluated in a different coordinate system $C_{[j,k]}(\phi', \psi', \ldots)$ will be denoted $C'_{[j,k]}$. Note that $|\, C_{[d,d]} \,|$ defines a new relative invariant of weight $-2m$.

If the matrix $C_{[j,k]}$ satisfies

$$C'_{[j,k]} = A_{[j]}\, C_{[j,k]}\, A_{[k]}^{t}$$

instead, it will be called a *contravariant matrix*, and if it satisfies

$$C'_{[j,k]} = A_{[j]}\, C_{[j,k]}\, A_{[k]}^{-1}$$

it will be called *left contravariant and right covariant*, with a similar definition for matrices which are *left covariant and right contravariant.* Clearly, the determinant of a square covariant matrix $C_{[d,d]}$ is a relative invariant of weight $2m$, and the eigenvalues of a square left covariant and right contravariant matrix are absolute invariants.

The simplest example of a square contravariant matrix, is $\Phi_{[d,d]}^{-1}$, which is generally well defined, unless the matrix $\Phi_{[d,d]}$ is singular. We have already encountered examples of matrices which are left covariant and right contravariant : $\Phi_{[d,d]}\Psi_{[d,d]}^{-1}$.

### 5.6. Computation of orthogonal invariants

If the coordinate transformations are restricted to Euclidean transformations, i.e. the matrix $A$ is orthogonal, the four kinds of matrices defined above coincide, and we only talk about covariant matrices. That is, since $A_{[d]}$ is orthogonal when $A$ is orthogonal, a matrix $C_{[j,k]}$ is covariant with respect to orthogonal transformations if it satisfies

$$C'_{[j,k]} = A_{[j]}\, C_{[j,k]}\, A_{[k]}^{t}\,.$$

If $C_{[j,k]}$ is also square, with $j = k = d$, then its $h_d$ eigenvalues are orthogonal invariants, because in this case the matrix $C_{[d,d]} - \theta I$ is also covariant for every value of $\theta$, and so the coefficients, or equivalently the roots, of the characteristic polynomial $|\, C_{[d,d]} - \theta I \,|$, are invariants. In particular

Corollary 5.5. *Let $\phi$ be a form of even degree $d = 2k$. Then, the coefficients of the characteristic polynomial $|\, \Phi_{[k,k]} - \theta\, I \,|$ are orthogonal invariants, or equivalently, the eigenvalues of the square matrix $\Phi_{[k,k]}$ are orthogonal invariants.*

Note that, from the computational point of view, computing eigenvalues is much less expensive than expanding the determinants needed to obtain the coefficients of the characteristic polynomials, and computing eigenvalues requires in the order of $n^3$ operations, where $n$ is the size of the square matrices involved.

In the orthogonal case we can also define a *norm* in the vector space of forms of degree $d$ which is invariant under orthogonal transformations. This norm will be used later to define the Euclidean center of a polynomial.

LEMMA 5.5. *Let* $\phi(x) = \Phi^t_{[d]}X_{[d]}(x)$ *be a form of degree* $d$. *Then,*

$$\|\phi\|^2 = \Phi^t_{[d]}\Phi_{[d]} = \sum_{\alpha} \frac{1}{\alpha!}\Phi^2_\alpha$$

*is an orthogonal invariant of* $\phi$.

PROOF. If $x' = Ax$ is an orthogonal transformation, and $\phi'(x') = \phi(x)$ is the corresponding form in the new coordinate system, then $\Phi'^t_{[d]}\Phi'_{[d]} = \Phi^t_{[d]}A^t_{[d]}A_{[d]}\Phi_{[d]} = \Phi^t_{[d]}\Phi_{[d]}$ because $A_{[d]}$ is orthogonal. □

With respect to join orthogonal invariants of two or more forms, we also have a stronger result.

LEMMA 5.6. *Let* $\phi$ *and* $\psi$ *be two forms of degree* $d = j+k$, *then the eigenvalues of the square matrix* $\Phi_{[k,j]}\Psi_{[j,k]}$ *are join orthogonal invariants of the pair.*

PROOF. It is sufficient to note that, if $x' = Ax$ is a coordinate transformation, then $\Phi_{[k,j]}\Psi_{[j,k]}$ is a covariant matrix :

$$\Phi'_{[k,j]}\Psi'_{[j,k]} = \left(A_{[k]}\Phi_{[k,j]}A^t_{[j]}\right)\left(A_{[j]}\Psi_{[j,k]}A^t_{[k]}\right) = A_{[k]}\left(\Phi_{[k,j]}\Psi_{[j,k]}\right)A^t_{[k]} ,$$

because $A_{[j]}$ is orthogonal. □

The previous result has the obvious extension to join invariants of three or more forms, but we leave it to the reader. Also, we can apply the previous lemma to only one form. That is, we take $\psi = \phi$, and the eigenvalues of the covariant matrix $\Phi_{[k,j]}\Phi_{[j,k]}$ are orthogonal invariants of the form $\phi$. We will use this construction in the definition of the intrinsic Euclidean orientation of a polynomial, in section 8.

A last technique to compute join invariants is based on combining several matrices of coefficients to build block matrices. The idea behind it is basically the same as in the previous cases, but with a little twist. Let's explain it for just three forms. The construction can be generalized very easily to more forms.

LEMMA 5.7. *Let* $\phi(x), \psi(x)$ *and* $\xi(x)$ *be three forms of degrees* $2j$, $j+k$ *and* $2k$ *respectively. Then, the eigenvalues of the block matrix*

$$\begin{pmatrix} \Phi_{[j,j]} & \Psi_{[j,k]} \\ \Psi_{[k,j]} & \Xi_{[k,k]} \end{pmatrix}$$

*are join orthogonal invariants of the three forms.*

PROOF. Just look at the transformation rules of the block matrix

$$\begin{pmatrix} \Phi'_{[j,j]} & \Psi'_{[j,k]} \\ \Psi'_{[k,j]} & \Xi'_{[k,k]} \end{pmatrix} = \begin{pmatrix} A_{[j]} & 0 \\ 0 & A_{[k]} \end{pmatrix} \begin{pmatrix} \Phi_{[j,j]} & \Psi_{[j,k]} \\ \Psi_{[k,j]} & \Xi_{[k,k]} \end{pmatrix} \begin{pmatrix} A_{[j]} & 0 \\ 0 & A_{[k]} \end{pmatrix}^t ;$$

note that the block matrix function of the orthogonal matrix $A$ is also orthogonal. □

Note that, from the computational point of view, computing eigenvalues is much less expensive than expanding the determinants needed to obtain the coefficients of the characteristic polynomials, and computing eigenvalues require in the order of $n^3$ operations, where $n$ is here the size of the square matrices involved.

## 6. Brief historical remarks on invariant theory

By the middle of the nineteenth century it was known that, if in the quadratic form

$$\phi(x_1, x_2) = \tfrac{1}{2}\Phi_{(2,0)}x_1^2 + \Phi_{(1,1)}x_1x_2 + \tfrac{1}{2}\Phi_{(0,2)}x_2^2 \,,$$

we make a homogeneous change of variables $x' = Ax$, where $A$ is a nonsingular $2 \times 2$ matrix, we obtain a new quadratic form

$$\phi'(x_1', x_2') = \tfrac{1}{2}\Phi'_{(2,0)}{x_1'}^2 + \Phi'_{(1,1)}x_1'x_2' + \tfrac{1}{2}\Phi'_{(0,2)}{x_2'}^2 \,,$$

and the function $\mathcal{I}(\phi) = \Phi_{(2,0)}\Phi_{(0,2)} - \Phi_{(1,1)}^2$ of the coefficients of the form $\phi$ satisfies the following identity $\mathcal{I}(\phi') = |A|^{-2}\mathcal{I}(\phi)$, where $|A|$ is the determinant of the matrix $A$, i.e. $\mathcal{I}(\phi)$ is a relative invariant of weight 2 (see Dieudonné, 1971; Dieudonné and Carrell, 1971).

The classical invariant theory of algebraic forms was developed in the nineteenth century by Boole (1841, 1842), Cayley (1845, 1889), Clebsh (1872), Gordan (1887), Hilbert (1890, 1893), Sylvester (1904), Grace and Young (1903), Elliot (1913) and others (Salmon, 1866; Dickson, 1914), to solve the problem of classification of projective algebraic varieties, i.e. sets of common zeros of several homogeneous polynomials. In this century, the main contributions have been by Weyl (1939), Mumford (1965) and others (Gurevich, 1964; Springer, 1977). The projective coordinate transformations define a relation of equivalence in the family of algebraic varieties, with two varieties being equivalent if one of them can be transformed into the other by a projective transformation. The classical approach to the classification problem, as for example the classification of planar algebraic curves defined by a single form $\phi(x)$ of degree $d$ in three variables, is to find a set of relative or absolute invariants, $\{\mathcal{I}_1(\phi), \mathcal{I}_2(\phi), \ldots\}$ whose values determine the class that the form belongs to. One naturally tries to find a minimal family, and Hilbert (Hilbert, 1890, 1893; Ackerman, 1978) proved that there exist a finite number of *polynomial* invariants, a *fundamental system* of invariants, such that every other polynomial invariant is equal to an algebraic combination of the members of the fundamental system. But Hilbert's proof is not constructive, and the problem is then, how to compute a fundamental system of polynomial invariants. Algorithms exist, such as the Straightening Algorithm (Rota and Sturmfels, 1989), which is the implementation of the symbolic method of the german school, but they are computationally expensive (White, 1989). It is important to emphasize that, although the symbolic method reduces the computation of all the polynomial invariants of one or more forms to the combinatorial problem of listing all the multilinear invariants of many vectors, the evaluation of the symbolic expressions is computationally much more expensive than the methods based on eigenvalues introduced in this paper. For example, the symbolic expression for the characteristic polynomial of a matrix, as a polynomial in the entries of the matrix, has

$n!$ terms, but computing the eigenvalues of the same matrix only requires in the order of $n^3$ operations. A detailed description of the symbolic method, using current notation and terminology, can be found in Weyl (1939) and Dieudonné (1971). For the complexity of matrix computations see Golub and Van Loan (1983). For a more detailed article on the history of invariant theory, see Parshall (1990).

## 7. Intrinsic Euclidean center

Our definition of the *intrinsic Euclidean center*, or just the *center*, of a 2D curve or 3D surface of degree $d \geq 2$ is a generalization to $d > 2$ of the well-known case of a nonsingular quadratic curve or surface. We write the polynomial $f$ of degree $d$ as a sum of forms

$$f(x) = \sum_{k=0}^{d} f_k(x) ,$$

where $f_k$ is a form of degree $k$, and $f_d \neq 0$. For example, a quadratic polynomial of two variables can be written as

$$\begin{aligned} f(x) &= f_2(x) + f_1(x) + f_0 \\ &= [\tfrac{1}{2}F_{(2,0)}\, x_1^2 + F_{(1,1)}\, x_1 x_2 + \tfrac{1}{2}F_{(0,2)}\, x_2^2] + [F_{(1,0)}\, x_1 + F_{(0,1)} x_2] + [F_{(0,0)}]. \end{aligned} \quad (7.1)$$

For every fixed space vector $y$, the polynomial $g(x) = f(x+y)$, as a polynomial in $x$, has exactly the same degree $d$, and so it can also be written in a unique way as a sum of forms

$$f(x+y) = g(x) = \sum_{k=0}^{d} g_k(x) ,$$

where the coefficients of the homogeneous polynomial $g_k$ are polynomials of degree $d-k$ in $y$. Particularly, the term of degree $d$ is invariant under translation

$$g_d \equiv f_d ,$$

and the term of degree $d-1$ is given by

$$g_{d-1} \equiv f_{d-1} + y^t \nabla f_d = f_{d-1} + \sum_{i=1}^{n} y_i \frac{\partial f_d}{\partial x_i} .$$

In the example of the quadratic polynomial of two variables (7.1), we have

$$g_1(x_1, x_2) = [F_{(1,0)} + F_{(2,0)}\, y_1 + F_{(1,1)}\, y_2 ]\, x_1 + [F_{(0,1)} + F_{(1,1)}\, y_1 + F_{(0,2)}\, y_2 ]\, x_2$$

We define the *center* of $f$ as the vector $y$ which minimizes the invariant norm (see Lemma 5.5 above) of the homogeneous polynomial $g_{d-1}$

$$\|f_{d-1} + y^t \nabla f_d\|^2 ,$$

a least squares problem, which has a unique solution if the vectors of coefficients of the partial derivatives of the term of degree $d$, the homogeneous polynomials

$$\frac{\partial f_d}{\partial x_1}, \ldots, \frac{\partial f_d}{\partial x_n}$$

are linearly independent. In the example of the quadratic polynomial in two variables, we have

$$\|g_1\|^2 = [F_{(1,0)} + F_{(2,0)}\, y_1 + F_{(1,1)}\, y_2]^2 + [F_{(0,1)} + F_{(1,1)}\, y_1 + F_{(0,2)}\, y_2]^2 ,$$

which yields the value zero at the point

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = -\begin{pmatrix} F_{(2,0)} & F_{(1,1)} \\ F_{(1,1)} & F_{(0,2)} \end{pmatrix}^{-1} \begin{pmatrix} F_{(1,0)} \\ F_{(0,1)} \end{pmatrix} \tag{7.2}$$

when the $2 \times 2$ matrix in the right side is nonsingular.

Using the matrix formulation of previous sections, if $f_k(x) = F^t_{[k]} X_{[k]}(x)$ for $k = 0, 1, \ldots, d$, it is not difficult to see that $Df_d(x) = F_{[d-1,1]} X_{[d-1]}(x)$. It follows that $g_{d-1}(x) = \left[F_{[d-1]} + F_{[d-1,1]} y\right]$, and so

$$\|g_{d-1}\|^2 = \left\|F_{[d-1]} + F_{[d-1,1]} y\right\|^2$$

where the norm on the right side is the Euclidean norm. The conditions for unique minimizer is now a rank constraint on the matrix $F_{[d-1,1]}$ which has to be full rank, or equivalently $F_{[1,d-1]} F_{[d-1,1]}$ has to be nonsingular. In general, the center of $f(x)$ is given explicitly by the following formula

$$y = -F^{\dagger}_{[d-1,1]} F_{[d-1]} ,$$

where $F^{\dagger}_{[d-1,1]}$ is the pseudoinverse of $F_{[d-1,1]}$, which, if the condition for single solution is satisfied, is equal to

$$\left[F_{[1,d-1]} F_{[d-1,1]}\right]^{-1} F_{[1,d-1]} .$$

Finally, we show that the intrinsic Euclidean center of a polynomial is a covariant vector.

LEMMA 7.1. *Let $f(x)$ be a polynomial of degree $d$, $x' = Ax + b$ an Euclidean coordinate transformation, $f'(x') = f(A^t(x' - b))$, $y = -F^{\dagger}_{[d-1,1]} F_{[d-1]}$ and $y' = -F'^{\dagger}_{[d-1,1]} F'_{[d-1]}$. Then $y' = Ay + b$.*

The intrinsic Euclidean center can also be defined when the partial derivatives of $f_d$ are not linearly independent, but due to lack of space we will omit the description of such extension.

## 8. Intrinsic Euclidean orientation

The *intrinsic Euclidean orientation* of a 2D curve or 3D surface can be defined in several ways, all of them based on the fact that a symmetric matrix with nonrepeated eigenvalues has an associated set of eigenvectors, thus generating $2^n$ different orthogonal coordinate systems having unit vectors in the directions of these eigenvectors. In the example of the quadratic polynomial of two variables, the eigenvectors of the matrix

$$\begin{pmatrix} F_{(2,0)} & F_{(1,1)} \\ F_{(1,1)} & F_{(0,2)} \end{pmatrix} \tag{8.1}$$

define the orientation of the polynomial, if its eigenvalues are not repeated.

For a polynomial $f=\sum_{k=0}^{d} f_k$ of degree $d$, we consider the symmetric $n \times n$ matrix $[F_{[1,d-1]}F_{[d-1,1]}]$ defined by the coefficients of the form of degree $d$. In the case of the quadratic polynomial of two variables, it is the square of (8.1), which has the same eigenvectors, if its eigenvalues are not repeated. If the matrix $[F_{[1,d-1]}F_{[d-1,1]}]$ has all different eigenvalues, we define the intrinsic Euclidean orientation of $f$ as the orientation induced by the eigenvectors of $[F_{[1,d-1]}F_{[d-1,1]}]$. Then, in order to disambiguate among the $2^n$ different frames of reference, we will find the location of certain fixed points, other than the center. Every nonzero component of a fixed point can be used to chose the orientation of the corresponding axis. If the matrix has repeated eigenvalues, we will have to use information provided by the other homogeneous terms of $f$. In general, we can consider the eigenvectors of the $n \times n$ matrix

$$\sum_{k=1}^{d} w_k F_{[1,k]}F_{[k,1]} \tag{8.2}$$

where $w_1, \ldots, w_d$ are fixed constants, chosen to minimize the likelihood of repeated eigenvalues among the family of expected curves or surfaces, where the polynomial $f = \sum_{k=0}^{d} f_k$ is assumed to have been previously centered. For every value of $w_1, \ldots, w_d$, the $n$ eigenvalues of (8.2) are Euclidean invariants of the polynomial $f$, and although they are not sufficient to differentiate between any two polynomials of the same degree, they can be used as the first step towards the classification of $f$.

The invariance of the intrinsic Euclidean orientation of a polynomial follows easily from the transformation properties of the covariance matrices of coefficients, and we omit the proof.

We have defined the intrinsic Euclidean orientation for a $d$-th degree algebraic 2D curve or 3D surface, solely in terms of the coefficients of its $d$-th degree monomials. Is this a stable representation ? This representation should be stable for two reasons. First, the coefficients of the highest degree monomials are significant in the polynomials that we use. They are significant for the interest regions because lower degree polynomials do not fit the data well there. They are significant when each polynomial is a product of a group of low degree polynomials that we use because, again, a single lower degree polynomial would not fit a group of surfaces involved. Second, the regions chosen to be interest regions are those for which the representative polynomials are not sensitive to small changes in the region used. The groups of low degree polynomial surfaces that we use are those that can be easily found, that is, for which the segmentation is very stable, even in the presence of partial occlusion.

## 9. Intrinsic Euclidean center and orientation of a 3D curve

An algebraic curve has been defined as the set of zeros of a vector $\mathbf{f}(x) = (f(x), g(x))^t$ of polynomials of degree $\leq d$, with at least one of the two components of degree $d$. We can decompose the polynomials as sums of forms

$$f = \sum_{i=0}^{d} f_i \qquad \text{and} \qquad g = \sum_{i=0}^{d} g_i \,,$$

and without loss of generality, we will assume that the degree of $f$ is $d$, and the two forms of higher degree, $f_d$ and $g_d$, are orthogonal with respect to the invariant inner product of forms of degree $d$. Otherwise, we can replace the two polynomials by two independent linear combinations of them

$$\begin{aligned} f' &= \lambda_{11} f + \lambda_{12} g \\ g' &= \lambda_{21} f + \lambda_{22} g \end{aligned}$$

which satisfy this condition. This transformation does not change the curve or surface. If the degree of $g$ is less than $d$, we define the center of the curve as the center of the surface associated with $f$.

If both $f$ and $g$ are polynomials of the same degree $d$, by using the same argument as above, we can assume that the invariant norms of $f_d$ and $g_d$ are both equal to 1. In this case we define the center as the point $y$ which minimizes the sum of the square norms of the two forms of degree $d-1$ of the translated polynomials, the quadratic

$$\| F_{[d-1]} + F_{[d-1,1]}\, y\|^2 + \| G_{[d-1]} + G_{[d-1,1]}\, y\|^2 \,,$$

which can also be written as

$$\| H_{[d-1]} + H_{[d-1,1]}\, y\|^2$$

where, for each pair of nonnegative integers $j$ and $k$, the matrix $H_{[j,k]}$ is constructed by concatenating the corresponding matrices of coefficients of $f$ and $g$

$$H_{[j,k]} = \begin{pmatrix} F_{[j,k]} \\ G_{[j,k]} \end{pmatrix} ,$$

and we also write $H_{[k]}$ instead of $H_{[k,0]}$. Finally, the solution is given by

$$y = -H^{\dagger}_{[d-1,1]} H_{[d-1]} \,.$$

when $[H_{[1,d-1]} H_{[d-1,1]}]$ is nonsingular, and can be extended to the singular case as well.

The ***intrinsic Euclidean orientation*** of an algebraic 3D curve can be defined in the same way, but using the matrices $H_{[j,k]}$ defined above, instead of the matrices $F_{[j,k]}$.

It can be proved that both the intrinsic center and orientation of a 3D curve defined in this way are independent of the coordinate system, but we omit the proof due to lack of space.

## 10. A Remark on algebraic curve and surface fitting

Since the homogeneous term $f_d$ of highest degree of a polynomial $f = \sum_{k=0}^{d} f_k$ of degree $d$ is invariant under translations, the invariant norm of $f_d$ is an Euclidean invariant of the polynomial $f$. This invariant can be used as a constraint for fitting an algebraic surface or 2D to a data set $\mathcal{D} = \{p_1, \ldots, p_q\}$, by minimizing the mean square error

$$\frac{1}{q} \sum_{i=1}^{q} |f(p_i)|^2 \,,$$

constrained by

$$\|f_d\|^2 = 1 \,.$$

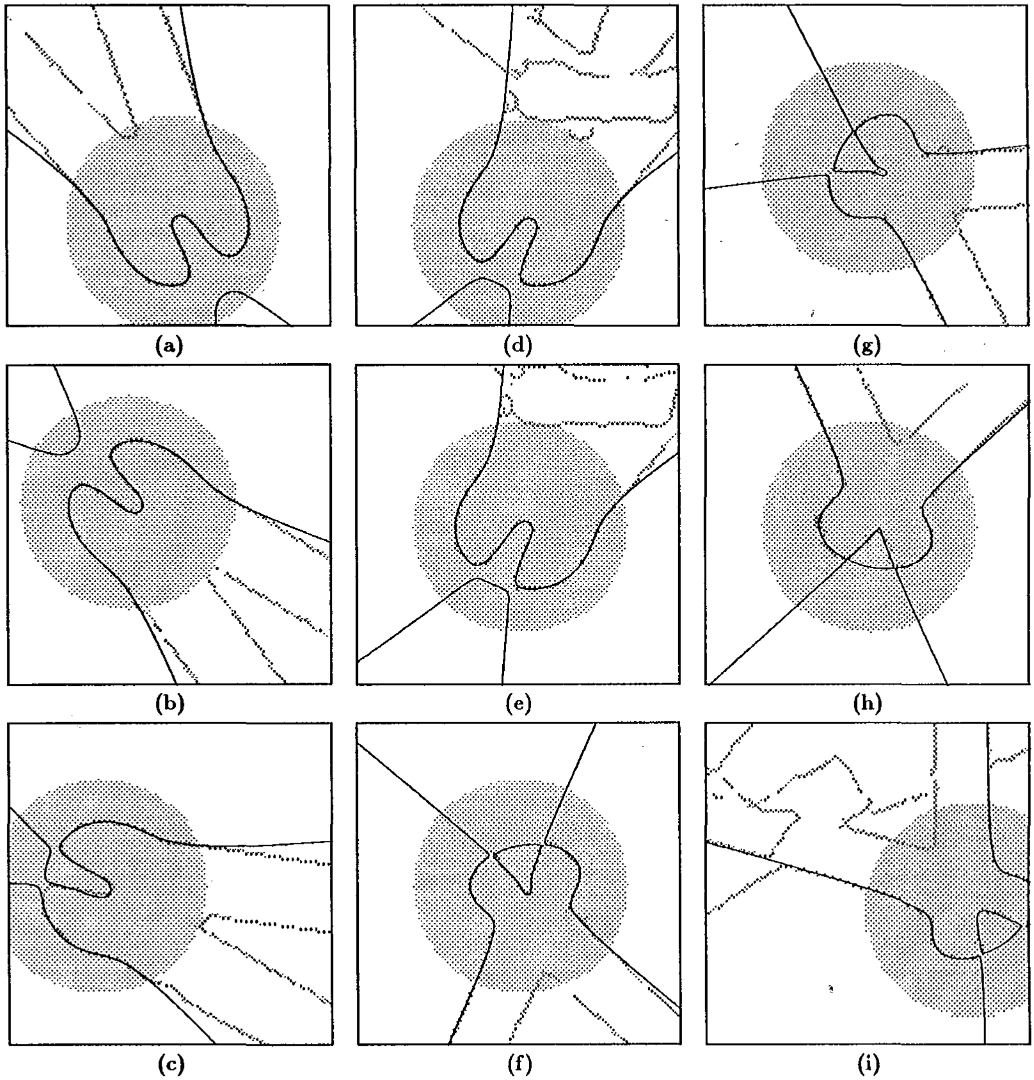


**Figure 8.** Fourth degree algebraic curve fits.

Since this constraint is invariant under Euclidean transformations, the curve or surface defined by the minimizer of the problem is independent of the coordinate system. Bookstein (1979) introduced the constraint

$$\|f_2\|^2 = \frac{1}{2}F^2_{(2,0)} + F^2_{(1,1)} + \frac{1}{2}F^2_{(0,2)} ,$$

for fitting conics to planar data sets following this method, and Cernuschi-Frias (1984) derived the constraint

$$\|f_2\|^2 = \frac{1}{2}F^2_{(2,0,0)} + F^2_{(1,1,0)} + F^2_{(1,0,1)} + \frac{1}{2}F^2_{(0,2,0)} + F^2_{(0,1,1)} + \frac{1}{2}F^2_{(0,0,2)} .$$

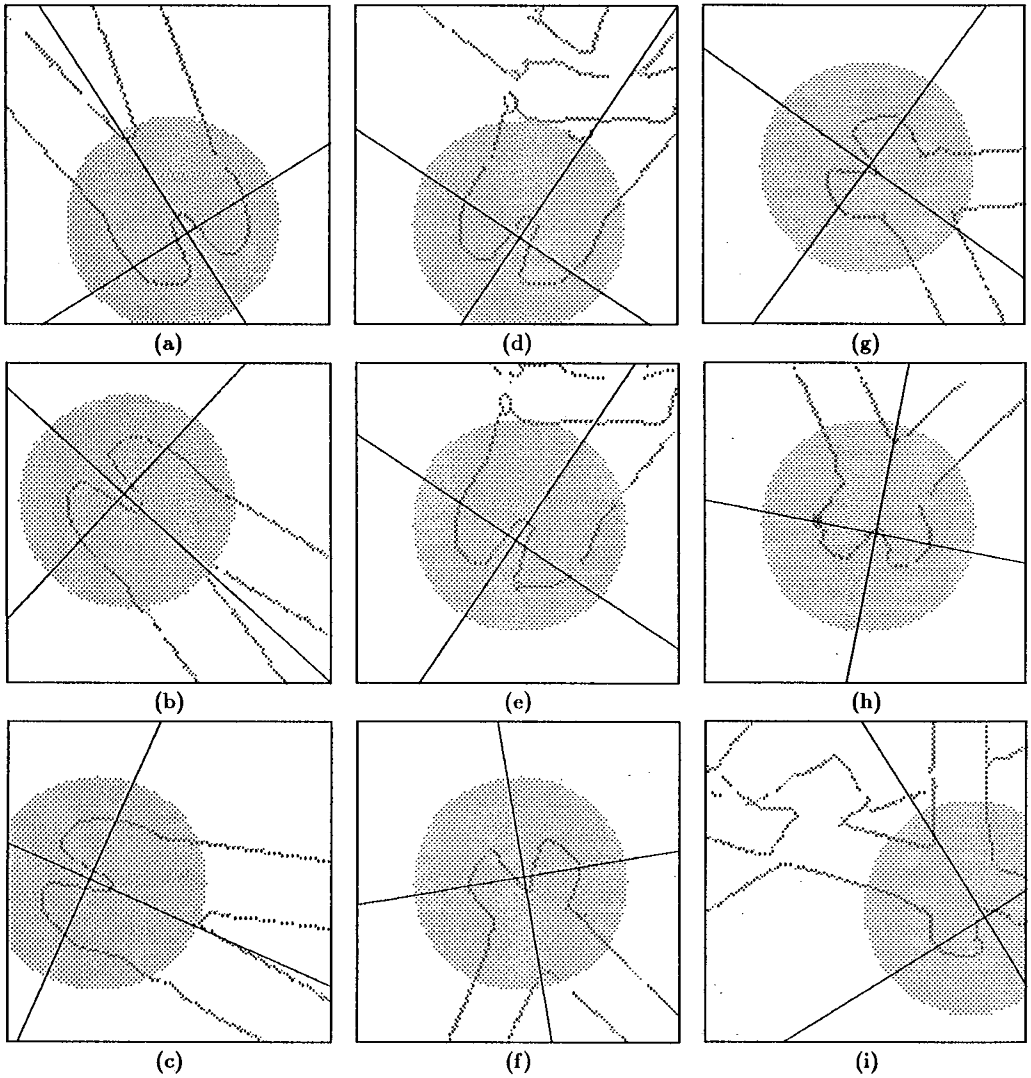

**Figure 9.** Intrinsic coordinate systems corresponding to the curves in figure 8.

for fitting quadric surfaces to three dimensional data sets. The problem with this approach to algebraic curve and surface fitting is that, in general, the mean square error is a very biased approximation of the mean square distance from the data points to the set of zeros of $f$. The curve or surface defined by the solution of this minimization problem, although invariant, fails to represent the data near singular points. The methods described in section 4 produce better results.

**Table 1.** Invariant vectors corresponding to the curves in figure 8.

| | | | | | |
|---|---|---|---|---|---|
| (a) | 3.04 | 9.31 | -2.49 | 3.42 | 11.22 |
| (b) | 2.98 | 9.33 | -2.59 | 3.17 | 11.27 |
| (c) | 3.93 | 8.97 | -3.76 | 3.82 | 10.73 |
| (d) | 3.95 | 8.96 | -3.76 | 3.89 | 10.70 |
| (e) | 4.03 | 8.92 | -3.90 | 3.87 | 10.66 |
| (f) | 5.64 | 8.00 | -6.24 | -4.06 | 9.40 |
| (g) | 6.27 | 7.52 | -7.07 | -4.28 | 8.69 |
| (h) | 5.92 | 7.80 | -6.55 | -4.25 | 9.10 |
| (i) | 5.14 | 8.33 | -5.58 | -4.21 | 9.74 |

**Table 2.** Distance matrix among invariant vectors corresponding to the curves in figure 8.

| | (a) | (b) | (c) | (d) | (e) | (f) | (g) | (h) | (i) |
|---|---|---|---|---|---|---|---|---|---|
| (a) | 0 | 3 | 17 | 17 | 19 | 90 | 100 | 95 | 87 |
| (b) | 3 | 0 | 18 | 18 | 20 | 88 | 98 | 93 | 85 |
| (c) | 17 | 18 | 0 | 1 | 2 | 86 | 94 | 90 | 84 |
| (d) | 17 | 18 | 1 | 0 | 2 | 87 | 95 | 91 | 85 |
| (e) | 19 | 20 | 2 | 2 | 0 | 86 | 93 | 90 | 84 |
| (f) | 90 | 88 | 86 | 87 | 86 | 0 | 14 | 6 | 10 |
| (g) | 100 | 98 | 94 | 95 | 93 | 14 | 0 | 8 | 23 |
| (h) | 95 | 93 | 90 | 91 | 90 | 6 | 8 | 0 | 15 |
| (i) | 87 | 85 | 84 | 85 | 84 | 10 | 23 | 15 | 0 |

## 11. Examples

Figure 8 shows nine fourth degree 2D curves fitted to the data inside the grey circles using the methods described in section 4.

Multiplying by the proper constants, we normalize the polynomials such that the Euclidean norm of their leading form be equal to one. If we write the leading form as

$$f_4(x_1, x_2) = \frac{1}{24}F_{(4,0)}x_1^4 + \frac{1}{6}F_{(3,1)}x_1^3x_2 + \frac{1}{4}F_{(2,2)}x_1^2x_2^2 + \frac{1}{6}F_{(1,3)}x_1x_2^3 + \frac{1}{24}F_{(0,4)}x_2^4 ,$$

then its Euclidean norm is

$$\|f_4\|^2 = \frac{1}{24}F_{(4,0)}^2 + \frac{1}{6}F_{(3,1)}^2 + \frac{1}{4}F_{(2,2)}^2 + \frac{1}{6}F_{(1,3)}^2 + \frac{1}{24}F_{(0,4)}^2 .$$

Then we construct the matrices of coefficients

$$F_{[1,3]} = \begin{pmatrix} \frac{1}{\sqrt{6}}F_{(4,0)} & \frac{1}{\sqrt{2}}F_{(3,1)} & \frac{1}{\sqrt{2}}F_{(2,2)} & \frac{1}{\sqrt{6}}F_{(1,3)} \\ \frac{1}{\sqrt{6}}F_{(3,1)} & \frac{1}{\sqrt{2}}F_{(2,2)} & \frac{1}{\sqrt{2}}F_{(1,3)} & \frac{1}{\sqrt{6}}F_{(0,4)} \end{pmatrix}$$

and

$$F_{[2,2]} = \begin{pmatrix} \frac{1}{2}F_{(4,0)} & \frac{1}{\sqrt{2}}F_{(3,1)} & \frac{1}{2}F_{(2,2)} \\ \frac{1}{\sqrt{2}}F_{(3,1)} & F_{(2,2)} & \frac{1}{\sqrt{2}}F_{(1,3)} \\ \frac{1}{2}F_{(2,2)} & \frac{1}{\sqrt{2}}F_{(1,3)} & \frac{1}{2}F_{(0,4)} \end{pmatrix} .$$

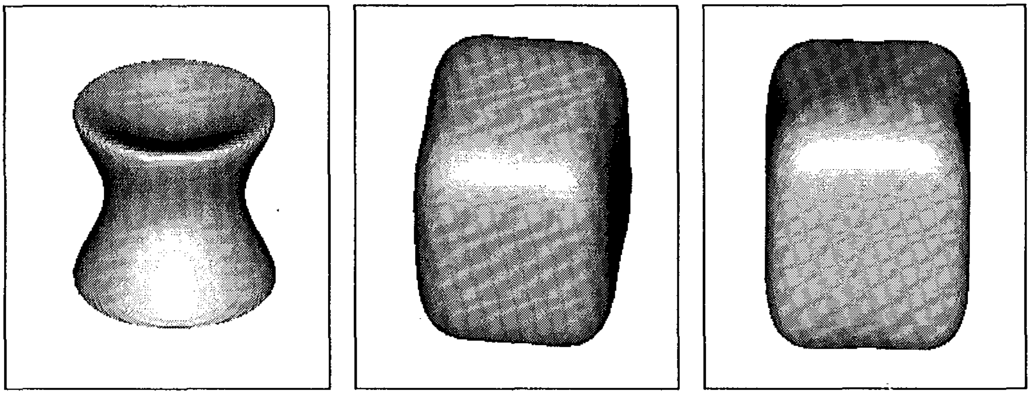


**Figure 10.** Fourth degree algebraic surfaces.

**Table 3.** Invariants corresponding to the surfaces in figure 10. Eigenvalues of the matrix $F_{[2,2]}$.

| | | | | | | |
|---|---|---|---|---|---|---|
| (a) | -.6600 | -.6600 | .7542 | .7542 | .8166 | 1.8233 |
| (b) | -.1775 | .0000 | .0000 | .5324 | 1.0575 | 2.1370 |
| (c) | -.1808 | .0000 | .0000 | .5262 | 1.0240 | 2.1334 |

For each one of the nine curves, we computed nine invariants of the leading forms. These vectors of invariants are shown in table 1.

The first two invariants are the square roots of the two eigenvalues of the symmetric positive definite $2\times 2$ matrix $F_{[1,3]}F_{[3,1]} = F_{[1,3]}F^t_{[1,3]}$, and the last three invariants are the three eigenvalues of the symmetric $2\times 2$ matrix $F_{[2,2]}$. The reason for taking the square roots is to maintain the five numbers within the same range, because $F_{[1,3]}F_{[3,1]}$ is a quadratic function of the coefficients, while $F_{[2,2]}$ is linear. The distances among the vectors of invariants, multiplied by ten and rounded to the closest integer, are shown in table 2. We can see in this table that these invariants very clearly separate the two classes.

Figure 9 shows the intrinsic coordinate systems corresponding to the same curves. Since the leading forms are nonsingular, the centers are given by

$$y = -F^{\dagger}_{[3,1]}F_{[3]} = -F^t_{[1,3]}\left[F_{[1,3]}F^t_{[1,3]}\right]^{-1}F_{[3]} ,$$

where $F_{[3]}$ is the vector of coefficients of the third degree form

$$F_{[3]} = \left( \tfrac{1}{\sqrt{6}}F_{(3,0)} \quad \tfrac{1}{\sqrt{2}}F_{(2,1)} \quad \tfrac{1}{\sqrt{2}}F_{(1,2)} \quad \tfrac{1}{\sqrt{6}}F_{(0,3)} \right)^t .$$

The intrinsic Euclidean orientation are determined by the eigenvectors of the matrix $F_{[1,3]}F_{[3,1]}$. Since the two eigenvalues of these matrices are well separated in the nine cases, the determination of the intrinsic Euclidean orientation based on these matrices is accurate.

Finally, figure 10 shows three fourth degree algebraic surfaces. The first surface, which is the same one shown in figure 2, is defined by the polynomial $x_1^4 - \frac{3}{4}x_1^2(x_2^2 + x_3^2) + (x_2^2+x_3^2)^2 - 1$. The second surface is defined by the polynomial $x_1^4 + 2x_2^4 - 2x_2^2x_3^2 + 4x_3^4 - 1$. The third surface was obtained from the second one through an orthogonal coordinate transformation, and so, should have the same invariants. This can be seen in table 2. The invariants shown in that table are the six eigenvalues of the matrices $F_{[2,2]}$ constructed with the coefficients of the leading forms.

## References


M. Ackerman (1978), "Hilbert's invariant theory papers", *Lie Groups: History, Frontiers and Applications*, vol. VIII, Math Science Press, Brookline, MA.

A. Albano (1974), "Representation of digitized contours in terms of conic arcs and straight-line segments", *Comput. Graph. Image Processing*, **3**, 23–33.

R. Bajcsy and F. Solina (1987), "Three dimensional object representation revisited", *Proc. 1st Int. Conf. Comput. Vision*, London, UK, 231–240.

D.H. Ballard (1981), "Generalizing the Hough transform to detect arbitrary shapes", *Patt. Recog.*, **13**, 111–122.

A.H. Barr (1981), "Superquadrics and angle-preserving transformations", *IEEE Comput. Graph. Appl.*, **1**, 11–23.

R.H. Biggerstaff (1972), "Three variations in dental arch form estimated by a quadratic equation", *J. Dental Research*, **51**, 1509.

R.M. Bolle, A. Califano, R. Kjeldsen and R.W. Taylor (1989a), "Visual recognition using concurrent and layered parameter networks", *Proc. IEEE Conf. Comput. Vision Patt. Recog.*, San Diego, CA, 625–631.

R.M. Bolle, A. Califano, R. Kjeldsen and R.W. Taylor (1989b), "Computer vision research at the IBM T.J. Watson Research Center", *Proc. DARPA Image Understanding Workshop*, Palo Alto, CA, 471–478.

R.M. Bolle and D.B. Cooper (1984), "Bayesian recognition of local 3D shape by approximating image intensity functions with quadric polynomials", *IEEE Trans. Patt. Anal. Mach. Intell.*, **6**, 418–429.

R.M. Bolle and D.B. Cooper (1986), "On optimally combining pieces of information, with applications to estimating 3D complex-object position from range data", *IEEE Trans. Patt. Anal. Mach. Intell.*, **8**, 619–638.

R.C. Bolles and P. Horaud (1986), "3DPO: a three-dimensional part orientation system", *Int. J. Robotics Research*, **5**, 3–26.

R.C. Bolles, P. Horaud and M.J. Hannah (1983), "3DPO: a three-dimensional part orientation system", *Proc. 8th Int. Joint Conf. Artif. Intell.*, Karlsruhe, Germany, 1116–1120.

F.L. Bookstein (1979), "Fitting conic sections to scattered data", *Comput. Vision Graph. Image Processing*, **9**, 56–71.

G. Boole (1841), "Exposition of a general theory of linear transformations", *Cambridge Math. J.*, **3**, 1–20.

G. Boole (1842), "Exposition of a general theory of linear transformations", *Cambridge Math. J.*, **3**, 106–119.

R.W. Brockett (1973), "Lie algebras and lie groups in control theory", *Geometrical Methods in System Theory*, D.Q. Mayne and R.W. Brockett, eds., Reidel Publishing Co., Dordrecht, The Netherlands, 43–82.

A. Cayley (1845), "On the theory of linear transformations", *Cambridge Math. J.*, **4**, 193–209.

A. Cayley (1889), *The Collected Mathematical Papers of Arthur Cayley, 1889–1897*, Cambridge University Press, Cambridge.

B. Cernuschi-Frias (1984), *Orientation and Location Parameter Estimation of Quadric Surfaces in 3D from a Sequence of Images*, PhD Thesis, Brown University, Providence, RI.

C.H. Chen and A.C. Kak (1989), "A robot vision system for recognizing 3D objects in low-order polynomial time", *IEEE Trans. Systems Man and Cybernetics*, **19**, 1535–1563.

D.S. Chen (1989), "A data-driven intermediate level feature extraction algorithm", *IEEE Trans. Patt. Anal. Mach. Intell.*, **11**, 749–758.

A. Clebsh (1872), *Theorie der Binären Algebraischen Formen*, B.G. Teubner, Leipzig.

D.B. Cooper and N. Yalabik (1975), *On the Cost of Approximating and Recognizing Noise-Perturbed Straight Lines and Quadratic Curve Segments in the Plane*, NASA-NSG-5036/I, Brown University, Providence, RI.

D.B. Cooper and N. Yalabik (1976), "On the computational cost of approximating and recognizing noise-

perturbed straight lines and quadratic arcs in the plane", *IEEE Trans. Comput.*, **25**, 1020–1032.
J.E. Dennis and R.B. Shnabel (1983), *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Prentice-Hall, NJ.
L.E. Dickson (1914), *Algebraic Invariants*, Math. Monographs, Vol 14, 1st edn., John Wiley & Sons, NY.
J.A. Dieudonné (1971), "La théorie des invariants au XIX$^e$ siècle", *Lecture Notes in Math. 244, Séminaire Bourbaki vol. 1970/71*, chapter Exposé n$^o$ 395, Springer-Verlag, 257–274.
J.A. Dieudonné and J.B. Carrell (1971), *Invariant Theory, Old and New*, Academic Press, NY and London.
R.O Duda and P.E. Hart (1973), *Pattern Classification and Scene Analysis*, John Wiley & Sons, NY.
E.B. Elliot (1913), *An Introduction to the Algebra of Quantics*, 2nd edn., Oxford University Press, Oxford.
O.D. Faugeras and M. Hebert (1983), "A 3D recognition and positioning algorithm using geometrical matching between primitive surfaces", *Proc. 8th Int. Joint Conf. Artif. Intell.*, Karlsruhe, Germany, 996–1002.
O.D. Faugeras and M. Hebert (1986), "The representation, recognition, and location of 3-D objects", *Int. J. Robotics Research*, **5**, 27–52.
O.D. Faugeras, M. Hebert and E Pauchon (1983), "Segmentation of range data into planar and quadric patches", *Proc. IEEE Conf. Comput. Vision Patt. Recog.*, Washington D.C., 8–13.
D. Forsyth, J.L. Mundy, A. Zisserman and C.M. Brown (1990), "Projectively invariant representation using implicit algebraic curves", *Proc. 1st European Conf. Comput. Vision*, Antibes, France, 427–436.
B.S. Garbow, J.M. Boyle, J.J. Dongarra and C.B. Moller (1977), *Matrix Eigensystem Routines - EISPACK Guide Extension*, Lecture Notes in Comput. Sci. 51, Springer-Verlag, NY.
D.B. Gennery (1980), "Object detection and measurement using stereo vision", *Proc. DARPA Image Understanding Workshop*, 161–167.
R. Gnanadesikan (1977), *Methods for Statistical Data Analysis of Multivariate Observations*, John Wiley & Sons, NY.
G. Golub and C.F. Van Loan (1983), *Matrix Computations*, John Hopkins University Press, Maryland.
P. Gordan and G. Kerschensteiner (1887), *Vorlensungen über Invariantentheorie*, Liepzig.
J.H. Grace and A. Young (1903), *The Algebra of Invariants*, Cambridge University Press, Cambridge.
W.E. Grimson (1989), "On the recognition of curved objects", *IEEE Trans. Patt. Anal. Mach. Intell.*, **11**, 632–643.
W.E. Grimson and T. Lozano-Perez (1984), "Model-based recognition and localization from sparse range or tactile data", *Int. J. Robotics Research*, **3**, 3–34.
W.E. Grimson and T. Lozano-Perez (1987), "Localizing overlapping parts by searching the interpretation tree", *IEEE Trans. Patt. Anal. Mach. Intell.*, **9**, 469–482.
A.D. Gross and T.E. Boult (1988), "Error of fit measures for recovering parametric solids", *Proc. 2nd Int. Conf. Comput. Vision*, Tampa, FL, 690–694.
G.B. Gurevich (1964), *Foundations of the Theory of Algebraic Invariants*, P. Noordhoff Ltd., Groningen, The Netherlands.
E.L. Hall, J.B.K. Tio, C.A. McPherson and F.A. Sadjadi (1982), "Measuring curved surfaces for robot vision", *IEEE Comput.*, **15**, 42–54.
S. Helgason (1984), *Groups and Geometric Analysis*, Academic Press, NY and London.
D. Hilbert (1890), "Über die theorie der algebraischen formen", *Math. Ann.*, 473.
D. Hilbert (1893), "Über die vollen invariantensysteme", *Math. Ann.*, 313.
J. Hong and H.J. Wolfson (1988), "An improved model-based matching method using footprints", *Proc. Int. Conf. Patt. Recog.*, Rome, Italy, 72–78.
E. Kishon and H. Wolfson (1987), "3D curve matching", *Proc. Workshop on Spatial Reasoning and Multi-Sensor Fusion*, Los Altos, CA, 250–261.
Y. Lamdan, J.T. Schwartz and H.J. Wolfson (1988), "Object recognition by affine invariant matching", *Proc. IEEE Conf. Comput. Vision Patt. Recog.*, Ann Arbor, Michigan, 335–344.
Y. Lamdan and H.J. Wolfson (1988), "Geometric hashing: a general and efficient model-based recognition scheme", *Proc. 2nd Int. Conf. Comput. Vision*, Tampa, FL, 238–249.
K. Levenberg (1944), "A method for the solution of certain problems in least squares", *Quarterly of Appl. Math.*, **2**, 164–168.
D. Marquardt (1963), "An algorithm for least-squares estimation of nonlinear parameters", *SIAM J. Appl. Math.*, **11**, 431–441.
J.J. Moré, B.S. Garbow and K.E. Hillstrom (1980), *User Guide for Minpack-1*, ANL-80-74, Argonne National Laboratories.
D. Mumford (1965), *Geometric Invariant Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol 34, Springer-Verlag, NY.

K.V.H. Parshall (1990), "The one-hundredth anniversary of the death of invariant theory ?", *Math. Intelligencer*, **12**(4), 10–16.
K.A. Paton (1970a), "Conic sections in chromosome analysis", *Patt. Recog.*, **2**, 39.
K.A. Paton (1970b), "Conic sections in automatic chromosome analysis", *Mach. Intell.*, **5**, 411.
J. Ponce and D.J. Kriegman (1989), "On recognizing and positioning curved 3D objects from image contours", *Proc. IEEE Workshop on Interpretation of 3D Scenes*, Austin, TX, 61–67.
V. Pratt (1987), "Direct least squares fitting of algebraic surfaces", *Comput. Graph.*, **21**, 145–152.
G.C. Rota and B. Sturmfels (1989), "Introduction to invariant theory in superalgebras", Invariant Theory and Tableaux, *The IMA Volumes in Math. and Its Applications*, **9**, Springer-Verlag, NY, 1–35.
G. Salmon (1866), *Modern Higher Algebra*, Hodges, Smith and Co., Dublin.
P.D. Sampson (1982), "Fitting conic sections to very scattered data: an iterative refinment of the Bookstein algorithm", *Comput. Vision Graph. Image Processing*, **18**, 97–108.
J. Schwartz and M. Sharir (1987), "Identification of partially obscured objects in two and three dimensions by matching noisy characteristic curves", *Int. J. Robotics Research*, **6**, 29–44.
F. Solina (1987), *Shape Recovery and Segmentation with Deformable Part Models*, PhD Thesis, University of Pennsylvania, PA.
T.A. Springer (1977), *Invariant Theory"* Lecture Notes in Math. 585, Springer-Verlag, NY.
B.T. Smith, J.M. Boyle, J.J. Dongarra, B.S. Garbow, Y. Ikebe, V.C. Klema and C.B. Moller (1976), *Matrix Eigensystem Routines - EISPACK Guide*, Lecture Notes in Comput. Sci. 6, Springer-Verlag, NY.
J.J. Sylvester (1904), *Collected Mathematical Papers, 1904–1912*, Cambridge University Press, Cambridge.
G. Taubin (1988a), *Algebraic Nonplanar Curve and Surface Estimation in 3-Space with Applications to Position Estimation*, Technical Report LEMS-43, Brown University, Providence, RI. Also, IBM Technical Report RC-13873.
G. Taubin (1988b), "Nonplanar curve and surface estimation in 3-space", *Proc. IEEE Conf. Robotics and Automation*, Philadelphia, 644–645.
G. Taubin (1989), *About Shape Descriptors and Shape Matching*, Technical Report LEMS-57, Brown University, Providence, RI.
G. Taubin (1990a), *Estimation of Planar Curves, Surfaces and Nonplanar Space Curves Defined by Implicit Equations, with Applications to Edge and Range Image Segmentation*, Technical Report LEMS-66, Brown University, Providence, RI. Also, *IEEE Trans. Patt. Anal. Mach. Intell.*, 1991.
G. Taubin (1991), *Recognition and Positioning of Rigid Object Using Algebraic and Moment Invariants*, PhD Thesis, Brown University, Providence, RI.
K. Turner (1974), *Computer Perception of Curved Objects using a Television Camera*, PhD Thesis, University of Edinburgh, Edinburgh.
R. Walker (1950), *Algebraic Curves*, Princeton University Press, Princeton, NJ.
H. Weyl (1939), *The Classical Groups*, Princeton University Press, Princeton, NJ.
N. White (1989), "Implementation of the straightening algorithm", Invariant Theory and Tableaux, *The IMA Volumes in Math. and Its Applications*, **19**, Springer-Verlag, NY, 36–45.
H. Wolfson (1990), "On curve matching", *IEEE Trans. Patt. Anal. Mach. Intell.*, **12**, 483–489.

## Appendix: Proofs

PROOF. (Lemma 5.2) The *multinomial formula* is

$$\tfrac{1}{d!}\,(x_1+\cdots+x_n)^d \;=\; \sum_{|\alpha|=d} \tfrac{1}{\alpha!}\, x_1^{\alpha_1}\cdots x_n^{\alpha_n} \;=\; \sum_{|\alpha|=d} \tfrac{1}{\alpha!}\, x^\alpha \,.$$

Let $x$ and $y$ be two $n$-dimensional vectors, and let us consider the multinomial expansion of the $d$-th power of the inner product $y^t x$ , the polynomial of $2n$ variables

$$\begin{aligned} \tfrac{1}{d!}\,(y^t x)^d &= \tfrac{1}{d!}\,(y_1x_1+\cdots+y_nx_n)^d \\ &= \textstyle\sum_{|\alpha|=d}\tfrac{1}{\alpha!}\,(y_1x_1)^{\alpha_1}\cdots(y_nx_n)^{\alpha_n} \;=\; \sum_{|\alpha|=d}\tfrac{1}{\alpha!}\,y^\alpha x^\alpha \,. \end{aligned}$$

This polynomial is homogeneous of degree $d$ in both $x$ and $y$ , and it is obviously invariant under simultaneous orthogonal transformations of the variables $x$-$y$ . In vector form,

$$\tfrac{1}{d!}\,(y^t x)^d \;=\; X_{[d]}(y)^t X_{[d]}(x) \,.$$

1-(a)) Let $A$ and $B$ be $n\times n$ nonsingular matrices. Then, the following expression

$$\begin{aligned}(AB)_{[d]}X_{[d]}(x) &= X_{[d]}((AB)x) = X_{[d]}(A(Bx)) = A_{[d]}X_{[d]}(Bx) \\ &= A_{[d]}(B_{[d]}X_{[d]}(x)) = (A_{[d]}B_{[d]})X_{[d]}(x)\end{aligned}$$

is a polynomial identity, and all the coefficients of the polynomials on the left side are identically to the corresponding coefficients of the polynomials on the right side, that is

$$(AB)_{[d]} = (A_{[d]}B_{[d]}) .$$

1-(b)) Follows from the uniqueness of representation of a homogeneous polynomial as a linear combination of monomials (5.2).

1-(c)) From (1-(b)), the identity matrix is map to the identity matrix. Let $A$ be a $n\times n$ nonsingular matrix. Apply (1-(a)) with $B = A^{-1}$ to obtain

$$I = (AA^{-1})_{[d]} = A_{[d]}(A^{-1})_{[d]} \quad\Rightarrow\quad (A_{[d]})^{-1} = (A^{-1})_{[d]} .$$

2) Let $A$ be a $n\times n$ nonsingular matrix. Then, the following expression

$$\begin{aligned}0 &= \tfrac{1}{d!}[((Ay)^t x)^d - (y^t(A^t x))^d] \\ &= X_{[d]}(Ay)^t X_{[d]}(x) - X_{[d]}(y)X_{[d]}(A^t x) = X_{[d]}(y)^t \left((A_{[d]})^t - (A^t)_{[d]}\right) X_{[d]}(x)\end{aligned}$$

is a polynomial identity, and all the coefficients of the polynomial on the right side are identically zero, that is

$$(A^t)_{[d]} = (A_{[d]})^t .$$

If $A$ is symmetric, we have

$$(A_{[d]})^t = (A^t)_{[d]} = A_{[d]} .$$

If the matrix $A$ is symmetric positive definite, we can write $A = BB^t$, for certain nonsingular $n\times n$ matrix $B$. Then

$$A_{[d]} = (BB^t)_{[d]} = B_{[d]}B^t_{[d]}$$

and so $A_{[d]}$ is positive definite as well. If $A$ is orthogonal, we have

$$(A_{[d]})^{-1} = (A^{-1})_{[d]} = (A^t)_{[d]} = (A_{[d]})^t .$$

3) If $\alpha$ and $\beta$ are two multiindices of size $d$, the $(\alpha,\beta)$-th. element of the matrix $A_{[d]}$ is

$$\sqrt{\tfrac{1}{\alpha!\,\beta!}}\, D^\beta((Ax)^\alpha) ,$$

Where $D^\beta$ is the partial differential operator

$$D^\beta = \left(\frac{\partial}{\partial x_1}\right)^{\beta_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\beta_n} .$$

If $\beta$ follows $\alpha$ in the lexicographical order, then, for certain $1 < k < n$ we have

$$\alpha_1 = \beta_1, \ldots, \alpha_{k-1} = \beta_{k-1}, \alpha_k > \beta_k ,$$

and so

$$\alpha_{k+1} + \cdots + \alpha_n < \beta_{k+1} + \cdots + \beta_n .$$

Since the matrix $A$ is lower triangular, the degree of

$$(Ax)^\alpha = \prod_{i=1}^{n} \left( \sum_{j=1}^{i} a_{ij}\, x_j \right)^{\alpha_i}$$

as a polynomial in $x_{k+1}, \ldots, x_n$ with coefficients polynomials in $x_1, \ldots, x_k$ is clearly not greater than $\alpha_{k+1} + \cdots + \alpha_n$, and so

$$\left( \frac{\partial}{\partial x_{k+1}} \right)^{\beta_{k+1}} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\beta_n} ((Ax)^\alpha) = 0 .$$

It follows that $D^\beta((Ax)^\alpha) = 0$, and the matrix $A_{[d]}$ is lower triangular.

4) For every matrix $A$, there exist an orthogonal matrix $Q$, and a lower triangular matrix $L$ such that $A = LQ$. Since the map $A \mapsto A_{[d]}$ is a homomorphism, we have $A_{[d]} = L_{[d]}Q_{[d]}$, where $L_{[d]}$ is lower triangular and $Q_{[d]}$ is orthogonal, i.e. the decomposition is preserved. Since $|A_{[d]}| = |L_{[d]}|$, without loss of generality we will assume that $A$ is lower triangular itself.

Now note that for every $1 \le k \le n$ the variable $x_k$ appears only in the last term of the product

$$\prod_{i=1}^{k} \left( \sum_{j=1}^{i} a_{ij}\, x_j \right)^{\alpha_i} ,$$

and so

$$\left( \frac{\partial}{\partial x_k} \right)^{\alpha_k} \left( \prod_{i=1}^{k} \left( \sum_{j=1}^{i} a_{ij}\, x_j \right)^{\alpha_i} \right) = \left( \prod_{i=1}^{k-1} \left( \sum_{j=1}^{i} a_{ij}\, x_j \right)^{\alpha_i} \right) \alpha_k!\, a_{kk}^{\alpha_k} .$$

By induction in $k = n, n-1, \ldots, 1$, it follows that the $\alpha$-th element of the diagonal of $A_{[d]}$ is

$$\tfrac{1}{\alpha!} D^\alpha((Ax)^\alpha) = a_{11}^{\alpha_1} \cdots a_{nn}^{\alpha_n} = a^\alpha$$

Since $A$ is triangular, $|A| = a_{11} \cdots a_{nn}$, and we have

$$|A_{[d]}| = \prod_{|\alpha|=d} a^\alpha = a^\gamma ,$$

where $\gamma = \sum_{|\alpha|=d} \alpha$. By symmetry, all the components of the multiindex $\gamma$ are equal, and so, for every $1 \le i \le n$

$$\gamma_i = \sum_{|\alpha|=d} \alpha_i = \tfrac{1}{n} \sum_{i=1}^{n} \sum_{|\alpha|=d} \alpha_i = \sum_{|\alpha|=d} = \tfrac{d}{n} \binom{n+d-1}{n-1} = \binom{n+d-1}{n} = m .$$

Finally

$$|A_{[d]}| = \left( \prod_{i=1}^{n} a_{ii} \right)^m = |A|^m .$$

□

PROOF. (Lemma 5.3) Let $D = (\partial/\partial x_1, \ldots, \partial/\partial x_n)^t$ be the vector of first order partial derivatives. For every multiindex $\alpha$ let

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} = \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} .$$

For every form of degree $d$

$$\psi(x) = \sum_{|\eta|=d} \tfrac{1}{\eta!}\, \Psi_\eta\, x^\eta$$

there is a corresponding homogeneous linear differential operator

$$\psi(D) = \sum_{|\eta|=d} \tfrac{1}{\eta!}\, \Psi_\eta\, D^\eta ,$$

and every homogeneous linear differential operator of degree $d$ can be written in this form in a unique way, i.e. the vector space of linear differential operators of order $d$ is a vector space of the same dimension $h_d$, and the map $\psi(x) \mapsto \psi(D)$ defines an isomorphism of vector spaces. For integers $j$ and $k$, the vector of differential operators $X_{[k]}(D)$ will be denoted $D_{[k]}$, and the matrix $X_{[j,k]}(D)$ will be denoted $D_{[j,k]}$. If $\phi$ is a form of degree $d$, from the Taylor formula $\Phi_\alpha = D^\alpha \phi$ for every multiindex $\alpha$, where $\{\Phi_\alpha : |\alpha| = d\}$ are the coefficients of $\phi$. It follows by construction that $\Phi_{[d]} = D_{[d]}\phi$, and $\Phi_{[j,k]} = D_{[j,k]}\phi$, for every pair of indices such that $j + k = d$.

If $x' = Ax$ is a coordinate transformation, the partial derivatives with respect to the two different coordinate systems are related by the chain rule

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial x'_j}{\partial x_i}\frac{\partial}{\partial x'_j} = \sum_{j=1}^{n} A_{ji}\frac{\partial}{\partial x'_j} ,$$

or in matrix form, $D' = (\partial/\partial x'_1, \ldots, \partial/\partial x'_n)^t = A^{-t}D$. It follows that

$$D'_{[k]} = X_{[k]}(D') = X_{[k]}(A^{-t}D) = (A^{-t})_{[k]}X_{[k]}(D) = A^{-t}_{[k]}D_{[k]} , \tag{11.1}$$

and

$$\begin{aligned} D'_{[k,j]} &= X_{[k]}(D')X^t_{[j]}(D') &&= X_{[k]}(A^{-t}D)X^t_{[j]}(A^{-t}D) = \\ &= A^{-t}_{[k]}X_{[k]}(D)X^t_{[j]}(D)A^{-1}_{[j]} &&= A^{-t}_{[k]}D_{[k,j]}A^{-1}_{[j]} . \end{aligned} \tag{11.2}$$

Finally, from (11.1)

$$\Phi'_{[d]} = D'_{[d]}\phi'(x') = A^{-t}_{[d]}D_{[d]}\phi(x) = A^{-t}_{[d]}\Phi_{[d]} ,$$

and if $d = j + k$, from (11.2)

$$\Phi'_{[j,k]} = D'_{[j,k]}\phi'(x') = A^{-t}_{[j]}\left(D_{[j,k]}\phi(x)\right)A^{-1}_{[k]} = A^{-t}_{[j]}\Phi_{[j,k]}A^{-1}_{[k]} .$$

□

PROOF. (Lemma 7.1) It is sufficient to consider the cases of pure translation and pure rotation separately. Let us first consider the case of a pure translation, i.e. $x' = x + b$. Since the term of degree $d$ of a polynomial is independent of the transformation parameters, we have

$$f'_d(x') = f_d(x) ,$$

or equivalently,

$$F'_{[d-1,1]} = F_{[d-1,1]} \, .$$

The term of degree $d-1$ is given by

$$f'_{d-1}(x') = f_{d-1}(x) - D_{[1]} f_{[d-1}(x) b \, ,$$

or, in terms of the coefficients,

$$F'_{[d-1]} = F_{[d-1]} - F_{[d-1,1]} b \, .$$

The center is in this case

$$y' = -{F'_{[d-1,1]}}^{\dagger} F'_{[d-1]} = -F^{\dagger}_{[d-1,1]} F_{[d-1]} + F^{\dagger}_{[d-1,1} F_{[d-1,1]} b = y + b \, .$$

Now, the case of pure rotation $x' = Ax$. In this case, since the terms of different degrees transform independently of each other, we can apply the transformation rules studied in the previous chapter

$$F'_{[d-1,1]} = A_{[d-1]} F_{[d-1,1]} A^t \qquad \text{and} \qquad F'_{[d-1]} = A_{[d-1]} F_{[d-1]} \, ,$$

and obtain

$${F'_{[d-1,1]}}^{\dagger} = A F^{\dagger}_{[d-1,1]} A^t_{[d-1]} \, .$$

Finally,

$$y' = -{F'_{[d-1,1]}}^{\dagger} F'_{[d-1]} = -A F^{\dagger}_{[d-1,1]} A^t_{[d-1]} A_{[d-1]} F_{[d-1]} = -A F^{\dagger}_{[d-1,1]} F_{[d-1]} = Ay \, .$$

□