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Chapter 9

A Mathematical Framework for Combinatorial/Structural Analysis of Linear Dynamical Systems by Means of Matroids

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This paper surveys some results on the structural analysis of linear dynamical systems using matroid-theoretic combinatorial methods. The mathematical model employed in this approach classifies the coefficients in the equations into independent physical parameters and dimensionless fixed constants. It is emphasized that relevant physical observations are crucial to successful mathematical modeling for structural analysis. In particular, the model is based on a kind of dimensional analysis. The concepts of the mixed matrix and its canonical form turn out to be convenient mathematical tools. An efficient algorithm for computing the canonical form is described in detail.

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1. Introduction

In many different fields of engineering, a graph-theoretic approach has proved to be useful for the analysis of large-scale systems, in which a huge number of elements are interconnected with each other. Graph-theoretic concepts are suitable for describing the interconnections and hence for analyzing those qualitative aspects of a system which result from the combinatorial structure of the system.

In control theory, to be specific, Lin (1974) discussed a system-theoretic property called controllability of a system in terms of the interconnection of the elements, introducing the concept of "structural controllability". In so doing Lin initiated the "structural approach" in control theory, which discusses the system-theoretic properties of a dynamical system based on combinatorial (mainly graph-theoretic) considerations. Typically, this approach starts with the state-space equations:

$$\dot{\mathbf{x}}(t) = \hat{A}\mathbf{x}(t) + \hat{B}\mathbf{u}(t)$$

and assumes that the nonzero entries of the matrices $\hat{A}$ and $\hat{B}$ are independent parameters. It has been gradually recognized, however, that this assumption is not sufficiently justifiable in many practical situations and, as a consequence, the graph-theoretic approach based on such a primitive mathematical model sometimes fails to be useful, as it ignores some important physical aspects.

This paper surveys the mathematical framework, proposed by the author, for the structural analysis of linear dynamical systems using a matroid-theoretic combinatorial method. This approach is based on a physically reasonable mathematical model, which makes it possible to describe real-world systems more faithfully than the primitive model employed in the graph-theoretic approach and which, at the same time, retains as much mathematical simplicity and convenience for the subsequent analysis. The mathematical model classifies the coefficients in the equations into independent physical parameters and dimensionless fixed constants, and describes dynamical systems by means of a class of structured polynomial matrices. It is emphasized that relevant physical observations are crucial to successful mathematical modeling for structural analysis. In particular, the model is based on a kind of dimensional analysis, and the physical-dimensional consistency is shown to have a significant implication for the algebraic or computational complexity of the resulting mathematical model. The following key words will represent some important features of the proposed model:

$$\begin{array}{rcl} \text{physical faith} & \longleftrightarrow & \text{mathematical convenience,} \\ \text{independent parameters} & \longleftrightarrow & \text{fixed constants,} \\ \text{dimensional consistency} & \longleftrightarrow & \text{algebraic/computational complexity.} \end{array}$$

The concepts of the mixed matrix and its canonical form turn out to be convenient mathematical tools. The fundamental properties of mixed matrices are described in section 3.1. Here are some of the nice properties enjoyed by a mixed matrix.

- The rank can be computed efficiently by a matroid-theoretic algorithm.
- A notion of irreducibility is defined with respect to a natural transformation (of physical significance).
- An irreducible component thus defined satisfies a number of nice properties that justify the name of irreducibility.
- There exists a canonical block-triangular decomposition into irreducible components called the combinatorial canonical form (CCF for short).

Section 3.2 affords a detailed description of the efficient algorithm for computing the canonical form which is only briefly explained in Murota (1987b).

Finally, in section 4 we shall show some properties of the structured polynomial matrices that are relevant to the structural analysis of dynamical systems using the proposed mathematical framework. These results answer system-theoretic questions such as controllability/observability, fixed modes in decentralized systems, disturbance decoupling, and structure at infinity of transfer matrices. The reader is referred to the author's research monograph (Murota, 1987b) for more information, unless otherwise indicated.

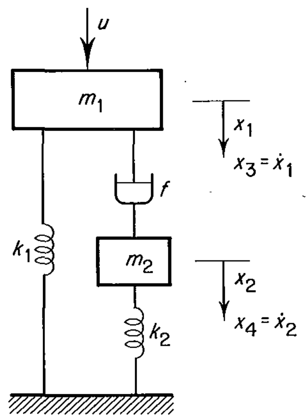

Figure 1. A mechanical system.

2. The Mathematical Model

Let us consider, as an example, a very simple mechanical system (see figure 1) which consists of two masses m_1, m_2, two springs k_1, k_2, and a damper f; u is the force exerted from outside. We may describe the system in the form of state-space equations (Kalman, 1963):

$$\dot{\mathbf{x}}(t) = \hat{A}\mathbf{x}(t) + \hat{B}\mathbf{u}(t) \tag{2.1}$$

in terms of $\mathbf{x} = (x_1, \ldots, x_4)$, where x_1 (resp. x_2) is the displacement of mass m_1 (resp. m_2), x_3 (resp. x_4) is its velocity and

$$\hat{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1/m_1 & 0 & -f/m_1 & f/m_1 \\ 0 & -k_2/m_2 & f/m_2 & -f/m_2 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{pmatrix}.$$

The state-space (2.1) has been useful for investigating analytic and algebraic properties of a dynamical system, and the structural or combinatorial analyses at the earlier stage (Lin, 1974) were also based on it. It has now been recognized, however, that the state-space equations are not very suitable for representing the combinatorial structure of a system, in that the entries of matrices $\hat{A}$ and $\hat{B}$ of (2.1) are usually not independent but interrelated to one another, being subject to algebraic relations. In this respect, the so-called descriptor form (Luenberger, 1977):

$$F\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{2.2}$$

or its Laplace transform:

$$sF\hat{\mathbf{x}}(s) = A\hat{\mathbf{x}}(s) + B\hat{\mathbf{u}}(s),$$

is more suitable. Then a system is described by a polynomial matrix

$$D(s) = (A - sF \mid B). \tag{2.3}$$

For our mechanical system it may be more natural to introduce two additional variables x_5 (= force by the damper f) and x_6 (= relative velocity of the two masses), where $x_5 = fx_6$ and $x_6 = \dot{x}_1 - \dot{x}_2$, and describe the system using the descriptor form (2.2). We then have

$$D(s) = \begin{pmatrix} -s & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -s & 0 & 1 & 0 & 0 & 0 \\ -k_1 & 0 & -sm_1 & 0 & -1 & 0 & 1 \\ 0 & -k_2 & 0 & -sm_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & f & 0 \\ -s & s & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

for the matrix of (2.3). Note that no complicated algebraic expressions are involved in this matrix.

The proposed mathematical model is based on two different physical observations; the first is the distinction between "accurate" and "inaccurate" numbers, and the second is the consistency with respect to physical dimensions.

The first observation is concerned with how we recognize the structure (or genericity) of a system. When a system is written in the form of (2.2) in terms of elementary variables, it is often justified to assume that the nonzero entries of the matrices F, A, etc. are classified into two groups: one group of generic parameters and another group of fixed constants. In other words, we can distinguish the following two kinds of numbers, which together characterize a physical system:

Inaccurate Numbers: Numbers representing independent physical parameters such as masses in mechanical systems and resistances in electrical networks which, being contaminated with noise and other errors, take values independent of one another, and therefore can be modeled as algebraically independent numbers, and

Accurate Numbers: Numbers accounting for various sorts of conservation laws such as Kirchhoff's laws which, stemming from topological incidence relations, are precise in value (often ± 1), and therefore cause no serious numerical difficulty in arithmetic operations on them.

We may also refer to the first kind of numbers as "system parameters" and to the second kind as "fixed constants". For our mechanical system, it will be natural to choose $\mathcal{T} = \{m_1, m_2, k_1, k_2, f\}$ as the set of system parameters; see Murota and Iri (1985) or Murota (1987b, Chapter 4) for further discussions in terms of examples.

This observation leads to the assumption that the matrices F, A and B in (2.2) are expressed as

$$F = Q_F + T_F, \qquad A = Q_A + T_A, \qquad B = Q_B + T_B,$$

where Q_F, Q_A and Q_B are matrices over $\mathbf{Q}$ (the field of rational numbers) and

(A1): The collection $\mathcal{T}$ of nonzero entries of T_F, T_A and T_B are algebraically independent over $\mathbf{Q}$.

As will be explained in section 3, such matrices as F, A and B are called mixed matrices. Accordingly, we express

$$D(s) = Q_D(s) + T_D(s) \tag{2.4}$$

with

$$Q_D(s) = (Q_A - sQ_F \mid Q_B), \qquad T_D(s) = (T_A - sT_F \mid T_B).$$

Then $Q_D(s)$ is a matrix over $\mathbf{Q}(s)$ (the field of rational functions in s with rational coefficients) and the nonzero entries of $T_D(s)$ are algebraically independent over $\mathbf{Q}(s)$.

For our mechanical system we have the decomposition (2.4) of $D(s)$ with

$$Q_D(s) = \begin{pmatrix} -s & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -s & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -s & s & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

and

$$T_D(s) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & -sm_1 & 0 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & -sm_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remark 2.1. It should be clear that assuming algebraic independence of $\mathcal{T}$ is equivalent to regarding the members of $\mathcal{T}$ as independent parameters, and therefore to considering the family of systems parametrized by those parameters in $\mathcal{T}$.

Remark 2.2. The rationality of the entries of Q_F, Q_A and Q_B is not essential. In case nonrational constants are involved, we may choose as $\mathbf{K}$ an appropriate extension field of $\mathbf{Q}$. The subfield $\mathbf{K}$ affects the computational complexity of algorithms.

The second physical observation due to Murota (1985) is a kind of dimensional analysis concerning the "accurate numbers", i.e. with $Q_D(s)$ in (2.4) (see also Murota, 1987a, 1987b, Chapter 4). The "accurate numbers" usually represent topological and/or geometrical incidence coefficients, which have no physical dimensions, so that it is natural to expect that the entries of Q_F, Q_A and Q_B are dimensionless constants. On the other hand, the indeterminate s in (2.3) should have the physical dimension of the inverse of time, since it corresponds to the differentiation with respect to time.

Since the system (2.2) is to represent a physical system, relevant physical dimensions are associated with both the variables $(\mathbf{x}, \mathbf{u})$ and the equations or, alternatively, with the columns and the rows of the matrix $D(s)$ of (2.3). Choosing time as one of the fundamental dimensions, we denote by $-c_j$ and $-r_i$ the exponent to the dimension of time associated respectively with the jth column and the ith row. The principle of dimensional homogeneity then demands that the (i,j) entry of $D(s)$ should have the dimension of time with exponent $c_j - r_i$.

Combining this fact with the observations on the nondimensionality of Q_F, Q_A and Q_B and on the dimension of s, we obtain

$$r_i - c_j = 1 \quad \text{if} \quad (Q_F)_{ij} \neq 0,$$

$$r_i - c_j = 0 \quad \text{if} \quad (Q_A)_{ij} \neq 0,$$

$$r_i - c_{n+j} = 0 \quad \text{if} \quad (Q_B)_{ij} \neq 0,$$

or in matrix form:

$$Q_D(s) = \operatorname{diag}[s^{r_1}, \ldots, s^{r_n}] \cdot Q_D(1) \cdot \operatorname{diag}[s^{-c_1}, \ldots, s^{-c_{n+m}}]. \tag{2.5}$$

This implies that

(A2): Every nonvanishing subdeterminant of $Q_D(s)$ is a monomial in s over $\mathbf{Q}$.

The converse is also true as stated below.

THEOREM 2.1. (MUROTA, 1985, 1987B) *Let $Q(s)$ be an $m \times n$ matrix with entries in $\mathbf{K}[s]$, where $\mathbf{K}\,(\supseteq \mathbf{Q})$ is a field and s an indeterminate over $\mathbf{K}$. Every nonvanishing subdeterminant of $Q(s)$ is a monomial in s over $\mathbf{K}$ iff*

$$Q(s) = \operatorname{diag}[s^{r_1}, \ldots, s^{r_m}] \cdot Q(1) \cdot \operatorname{diag}[s^{-c_1}, \ldots, s^{-c_n}]$$

for some integers r_i $(i = 1, \ldots, m)$ and c_j $(j = 1, \ldots, n)$.

For our mechanical system we may choose time T, length L and mass M as the fundamental quantities. Then the physical dimensions associated with the equations, i.e. with the rows of $D(s)$, are

$$T^{-1}L,\ T^{-1}L,\ T^{-2}LM,\ T^{-2}LM,\ T^{-2}LM,\ T^{-1}L,$$

whereas those with the variables (x_i and u), i.e. with the columns of $D(s)$, are

$$L,\ L,\ T^{-1}L,\ T^{-1}L,\ T^{-2}LM,\ T^{-1}L,\ T^{-2}LM.$$

We see that $Q_D(s)$ admits an expression of the form (2.5) if we choose the negative of the exponents to T as r_i and c_j, i.e.

$$r_1 = r_2 = 1,\ r_3 = r_4 = r_5 = 2,\ r_6 = 1;$$

$$c_1 = c_2 = 0,\ c_3 = c_4 = 1,\ c_5 = 2,\ c_6 = 1,\ c_7 = 2.$$

In this way, our physical observations have led us to a class of polynomial matrices $D(s)$ as a mathematical model representing the structure of linear dynamical systems. Namely, we consider a polynomial matrix $D(s)$ in indeterminate s over a field $\mathbf{F}(\supset \mathbf{Q})$ which is represented as

$$D(s) = Q_D(s) + T_D(s), \tag{2.6}$$

where

(A1): The nonzero coefficients $\mathcal{T}$ $(\subseteq \mathbf{F})$ of the entries of $T_D(s)$ are algebraically independent over $\mathbf{Q}$, and

(A2): Every nonvanishing subdeterminant of $Q_D(s)$ is a monomial in s over $\mathbf{Q}$.

Note that assumption (A1) implies that $D(s)$ is a mixed matrix (cf. section 3 for the definition) with respect to $\mathbf{K} = \mathbf{Q}(s)$. In section 4 we consider control-theoretic problems using such a mathematical model. It is noted, however, that $\mathcal{T}$ and $D(s)$ may be replaced by different objects depending on the problem.

3. Mixed Matrices

3.1. Fundamental properties

This section lists some known properties of a mixed matrix, a layered mixed matrix and their canonical forms, which constitute the mathematical foundation for the structural analysis based on the mathematical model of section 2. The notion of mixed matrix was introduced by Murota and Iri (1985); see Murota (1987b, 1990b) for the proofs.

For a matrix A, the row set and the column set of A are denoted by $\mathrm{Row}(A)$ and $\mathrm{Col}(A)$. For $I \subseteq \mathrm{Row}(A)$ and $J \subseteq \mathrm{Col}(A)$, $A[I, J]$ means the submatrix of A with row set I and column set J. The rank of A is written as $\mathrm{rank}\, A$. The (multi)set of nonzero entries of A is denoted by $\mathcal{N}(A)$. The zero/nonzero structure of a matrix A is represented by a bipartite graph $G(A) = (\mathrm{Row}(A), \mathrm{Col}(A), \mathcal{N}(A))$ with vertex set $\mathrm{Row}(A) \cup \mathrm{Col}(A)$ and arc set $\mathcal{N}(A)$. The term-rank of A is equal to the maximum size of a matching in $G(A)$.

Let $\mathbf{K}$ be a subfield of a field $\mathbf{F}$. A matrix A over $\mathbf{F}$ is called a *mixed matrix* with respect to $\mathbf{K}$ if

$$A = Q + T, \tag{3.1}$$

where

(i) $Q = (Q_{ij})$ is a matrix over $\mathbf{K}$, and
(ii) $T = (T_{ij})$ is a matrix over $\mathbf{F}$ such that the set $\mathcal{T} = \mathcal{N}(T)$ of its nonzero entries is (collectively) algebraically independent over $\mathbf{K}$.

Note that a polynomial matrix $D(s)$ of (2.6) satisfying assumption (A1) is a mixed matrix with respect to $\mathbf{K} = \mathbf{Q}(s)$.

The following identity is fundamental. It can be translated nicely into the matroid-theoretic language and enables us to compute the rank of A by an efficient algorithm using arithmetic operations in the subfield $\mathbf{K}$ only.

THEOREM 3.1. (Murota and Iri, 1985) *For a mixed matrix* $A = Q + T$,

$$\mathrm{rank}\, A = \max\{\mathrm{rank}\, Q[I, J] + \textit{term-rank}\, T[R - I, C - J] \mid I \subseteq R,\ J \subseteq C\},$$

where $R = \mathrm{Row}(A), C = \mathrm{Col}(A)$.

A matrix A is called a *layered mixed matrix* (or an *LM-matrix*) with respect to $\mathbf{K}$ if it takes the following form (possibly after a permutation of rows):

$$A = \begin{pmatrix} Q \\ T \end{pmatrix} \tag{3.2}$$

and Q and T of (3.2) meet the requirements (i) and (ii) above. In other words, an LM-matrix is a mixed matrix (3.1) such that the nonzero rows of Q-part and T-part are disjoint.

With an LM-matrix A of (3.2) we associate a function p as follows. Set $\mathrm{Row}(Q) = R_Q$, $\mathrm{Row}(T) = R_T$ and $\mathrm{Row}(A) = R$; then $R = R_Q \cup R_T$. The column sets of A, Q and T, being identified with one another, are denoted by C; namely, $\mathrm{Col}(A) = \mathrm{Col}(Q) =$

$\mathrm{Col}(T) = C$. Put

$$\begin{aligned} \rho(I,J) &= \operatorname{rank} Q[I,J], \quad I \subseteq R_Q, J \subseteq C, \\ \Gamma(I,J) &= \bigcup_{j\in J}\{i \in I \mid T_{ij} \neq 0\}, \quad I \subseteq R_T, J \subseteq C, \\ \gamma(I,J) &= |\Gamma(I,J)|, \quad I \subseteq R_T, J \subseteq C, \\ p(I,J) &= \rho(I \cap R_Q, J) + \gamma(I \cap R_T, J) - |J|, \quad I \subseteq R, J \subseteq C. \end{aligned} \tag{3.3}$$

The function $p : 2^R \times 2^C \to \mathbf{Z}$ is bisubmodular:

$$p(I_1 \cup I_2, J_1 \cap J_2) + p(I_1 \cap I_2, J_1 \cup J_2) \leq p(I_1, J_1) + p(I_2, J_2), \quad I_i \subseteq R, J_i \subseteq C \ (i = 1, 2).$$

Put

$$L(I) = \{J \subseteq C \mid p(I,J) \leq p(I,J'), \forall J' \subseteq C\}, \quad I \subseteq R. \tag{3.4}$$

For each $I \subseteq R$, $L(I)$ forms a sublattice of 2^C by virtue of the bisubmodularity of p.

Based on the Rank Identity in Theorem 3.1 we can prove the following, an extension of the well-known min-max characterization of the term rank of a matrix or the maximum matching in a bipartite graph, which is ascribed to J. Egerváry, D. König, P. Hall, R. Rado, O. Ore and others.

THEOREM 3.2. (MUROTA, 1987B; MUROTA *et al.*, 1987) *For an LM-matrix A,*

$$\operatorname{rank} A[I,J] = \min\{p(I,J') \mid J' \subseteq J\} + |J|, \quad I \subseteq R, \ J \subseteq C.$$

By the admissible transformation for an LM-matrix A of (3.2) we mean the transformation of the form:

$$P_{\mathrm{r}} \begin{pmatrix} S & O \\ O & I \end{pmatrix} \begin{pmatrix} Q \\ T \end{pmatrix} P_{\mathrm{c}}, \tag{3.5}$$

where S is a nonsingular matrix over the subfield $\mathbf{K}$, and P_{r} and P_{c} are permutation matrices. The admissible transformation brings an LM-matrix into another LM-matrix and two LM-matrices are said to be *LM-equivalent* iff they are connected by an admissible transformation. Note that the function $p_R \equiv p(R, \cdot) : 2^C \to \mathbf{Z}$ is an invariant under the LM-equivalence. That is, if A' is LM-equivalent to A, then $\mathrm{Col}(A')$ may be identified with $C = \mathrm{Col}(A)$ and the functions p and p' associated respectively with A and A' satisfy $p'(\mathrm{Row}(A'), J) = p(\mathrm{Row}(A), J)$ for $J \subseteq C$.

Remark 3.1. An electrical network is typically described by means of an LM-matrix when currents in and voltages across branches are chosen as the elementary variables (e.g. Iri, 1983; Recski, 1989). In that case, the Q-part represents the structural equations for Kirchhoff's current and voltage laws. As is well-known, there are a number of different ways of expressing these conservation laws. The LM-equivalence accounts exactly for the degree of freedom in expressing Kirchhoff's laws, as follows.

As an example, consider the simple electrical network of figure 2, which consists of five resistors (branches 1 to 5) and a voltage source (branch 6). Then the current ξ^i in and the voltage η_i across branch i should satisfy Kirchhoff's laws. To be specific, let us consider the current vector $\xi = (\xi^i \mid i = 1, \ldots, 6)$. If we translate Kirchhoff's current law

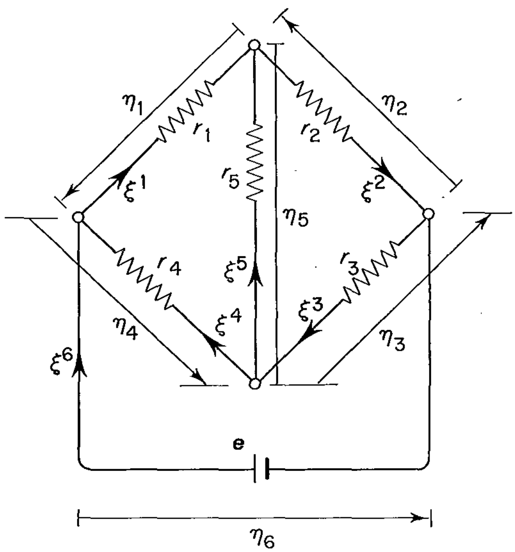

Figure 2. An electrical network.

into the statement that the algebraic sum of the currents flowing into each node is equal to zero, and notice that it is necessary and sufficient to consider three nodes, we obtain a mathematical expression $Q_1\xi = 0$ with

$$Q_1 = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{pmatrix}$$

for Kirchhoff's current law. We may obtain another equally natural mathematical expression if we pick up a tree in the underlying graph and consider the fundamental cutsets associated with it. For the tree consisting of the branches 1, 2 and 3, for instance, we have the expression $Q_2\xi = 0$ with

$$Q_2 = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{pmatrix}.$$

These two mathematical expressions for Kirchhoff's current law are not identical, having different coefficient matrices, but are equivalent in that they specify an identical subspace of the space of ξ. In fact, the coefficient matrices are related as $Q_1 = SQ_2$ with

$$S = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

This transformation matrix S corresponds to the second factor in the admissible transformation (3.5) of an LM-matrix.

It has been shown that there exists a finest block-triangular matrix, called the *combinatorial canonical form* (or *CCF* for short), among the matrices which are LM-equivalent to each other (cf. Theorem 3.3 below). Since the transformation (3.5) is more general than mere permutations of rows and columns, the CCF is a generalization of the canonical decomposition of a bipartite graph due to Dulmage and Mendelsohn (1959). An algorithm for the CCF is described in detail in section 3.2.

The CCF can be constructed based on the submodular function $p_R \equiv p(R,\cdot) : 2^C \to \mathbf{Z}$ of (3.3) and the lattice $L(R) \subseteq 2^C$ of (3.4) with the aid of a general principle, the Jordan-Hölder type decomposition principle, for a submodular function. It should be recalled that by Birkhoff's representation theorem (Aigner, 1979), a sublattice of 2^C is in one-to-one correspondence with a partition of C into partially ordered blocks $\{C_0; C_1, \ldots, C_b; C_\infty\}$, where

$$C_k \cap C_l = \emptyset \quad \text{if} \quad k \neq l, \{k,l\} \subseteq \{0,1,\ldots,b,\infty\},$$

and $C_k \neq \emptyset$ for $k = 1, \ldots, b$ (C_0 and C_∞ can be empty). The partial order among the blocks will be denoted as $\preceq$ and furthermore $C_k \prec C_l$ will mean that $C_k \preceq C_l, C_k \neq C_l$; and $C_k \prec\cdot\, C_l$ will mean that $C_k \prec C_l$ and there does not exist C_m such that $C_k \prec C_m \prec C_l$.

The following theorem claims for the existence of the CCF for an LM-matrix. It should be clear in the last statement (4) that the partitions of C into partially ordered blocks are partially ordered with respect to refinement relation.

THEOREM 3.3. (Murota, 1987b; Murota *et al.*, 1987) *For an LM-matrix A there exists an LM-matrix $\bar{A}$ which is LM-equivalent to A and satisfies the following properties.*

(1) $\bar{A}$ is block-triangularized, i.e.

$$\bar{A}[R_k, C_l] = O \quad \text{if} \quad 0 \leq l < k \leq \infty,$$

where $\{R_0; R_1, \ldots, R_b; R_\infty\}$ and $\{C_0; C_1, \ldots, C_b; C_\infty\}$ are partitions of $\mathrm{Row}(\bar{A})$ *and* $\mathrm{Col}(\bar{A})$ *respectively such that*

$$R_k \cap R_l = \emptyset, \; C_k \cap C_l = \emptyset \quad \text{if} \quad k \neq l, \{k,l\} \subseteq \{0,1,\ldots,b,\infty\},$$

and $R_k \neq \emptyset$, $C_k \neq \emptyset$ for $k = 1, \ldots, b$ (R_0, R_∞, C_0 and C_∞ can be empty).

Moreover, when $\mathrm{Col}(\bar{A})$ *is identified with* $\mathrm{Col}(A)$, *the partition $\{C_0; C_1, \ldots, C_b; C_\infty\}$ agrees with that defined by the lattice $L(R)$ and the partial order on $\{C_1, \ldots, C_b\}$ induced by the zero/nonzero structure of $\bar{A}$ agrees with the partial order $\preceq$ defined by $L(R)$; i.e.*

$$\begin{aligned} \bar{A}[R_k, C_l] &= O \quad \text{unless} \quad C_k \preceq C_l \quad (1 \leq k, l \leq b); \\ \bar{A}[R_k, C_l] &\neq O \quad \text{if} \quad C_k \prec\cdot\, C_l \quad (1 \leq k, l \leq b). \end{aligned}$$

(2)

$$\begin{aligned} |R_0| &< |C_0| \quad \text{if} \quad R_0 \neq \emptyset, \\ |R_k| &= |C_k| \quad (>0) \quad \text{for} \quad k = 1, \ldots, b, \\ |R_\infty| &> |C_\infty| \quad \text{if} \quad C_\infty \neq \emptyset. \end{aligned}$$

(3)

$$\begin{aligned} \operatorname{rank} \bar{A}[R_0, C_0] &= |R_0|, \\ \operatorname{rank} \bar{A}[R_k, C_k] &= |R_k| = |C_k| \quad \text{for} \quad k = 1, \ldots, b, \end{aligned}$$

$$\operatorname{rank} \bar{A}[R_\infty, C_\infty] \quad = \quad |C_\infty|.$$

(4) $\bar{A}$ is the finest block-triangular matrix with properties (2) and (3) that is LM-equivalent to A.

The matrix $\bar{A}$ above is the CCF of A. The submatrices $\bar{A}[R_0, C_0]$ and $\bar{A}[R_\infty, C_\infty]$ are called the *horizontal tail* and the *vertical tail*, respectively. An LM-matrix A will be called *LM-irreducible* or simply *irreducible* if its CCF does not split into more than one nonempty block, that is, if (a) $b = 1$ and $C_0 = R_\infty = \emptyset$, (b) $b = 0$ and $R_\infty = \emptyset$, or (c) $b = 0$ and $C_0 = \emptyset$. Each block $\bar{A}[R_k, C_k]$ of the CCF above is irreducible ($k = 0, 1, \ldots, b, \infty$).

Remark 3.2. The CCF is uniquely determined so far as the partitions of the row and column sets as well as the partial order among the blocks are concerned, whereas there remains some indeterminacy in the numerical values of the entries in the Q-part.

Example 3.1. Consider an LM-matrix $A = \binom{Q}{T}$ of (3.2) defined by

$$Q = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} -1 & 0 & 1 & -1 & 0 & 1 & 1 \\ -2 & 0 & 1 & -2 & 0 & 0 & 2 \\ 2 & 0 & 0 & 1 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

and

$$T = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} t_1 & 0 & 0 & 0 & 0 & t_2 & 0 \\ t_3 & 0 & 0 & 0 & t_4 & 0 & 0 \\ 0 & t_5 & 0 & t_6 & t_7 & 0 & 0 \\ 0 & t_8 & 0 & t_9 & t_{10} & 0 & t_{11} \end{pmatrix} \end{array}.$$

We have

$$\mathcal{T} = \mathcal{N}(T) = \{t_1, \cdots, t_{11}\}.$$

By choosing

$$S = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

in (3.5) we can transform Q to

$$Q' = SQ = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array},$$

which in turn yields the CCF:

$$
\bar{A} = P_{\mathrm{r}} \begin{pmatrix} Q' \\ T \end{pmatrix} P_{\mathrm{c}} =
\begin{array}{c}
\begin{array}{ccccccc} x_2 & x_4 & x_7 & x_3 & x_6 & x_1 & x_5 \end{array} \\
\begin{pmatrix}
0 & 1 & -1 & 0 & 1 & 1 & 0 \\
t_5 & t_6 & 0 & 0 & 0 & 0 & t_7 \\
t_8 & t_9 & t_{11} & 0 & 0 & 0 & t_{10} \\
 & & & 1 & 2 & 0 & 0 \\
 & & & & t_2 & t_1 & 0 \\
 & & & & & 1 & 1 \\
 & & & & & t_3 & t_4
\end{pmatrix}
\end{array}
$$

with

$$
P_{\mathrm{r}} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}, \quad
P_{\mathrm{c}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

The columns of $\bar{A}$ are partitioned into four blocks as

$$C_1 = \{x_2, x_4, x_7\},\ C_2 = \{x_3\},\ C_3 = \{x_6\},\ C_4 = \{x_1, x_5\}$$

with the partial order

$$
\begin{array}{ccccc}
 & & C_4 & & \\
 & & \downarrow & & \\
 & & C_3 & & \\
 & \swarrow & & \searrow & \\
C_1 & & & & C_2
\end{array}
$$

□

The irreducibility of an LM-matrix is characterized by the function p of (3.3) as follows.

THEOREM 3.4. *Let A be an LM-matrix with $R = \mathrm{Row}(A)$ and $C = \mathrm{Col}(A)$.*

(a) In case $|R| = |C|$ (> 0):

$$A \textit{ is irreducible} \iff p(R, J) > p(R, \emptyset) = p(R, C)\ (= 0),\ \forall J \neq \emptyset, C\ (J \subseteq C);$$

(b) In case $|R| < |C|$:

$$A \textit{ is irreducible} \iff p(R, J) > p(R, C),\ \forall J \neq C\ (J \subseteq C);$$

(c) In case $|R| > |C|$:

$$A \textit{ is irreducible} \iff p(R, J) > p(R, \emptyset)\ (= 0),\ \forall J \neq \emptyset\ (J \subseteq C).$$

Theorem 3.5 below states some properties of a square irreducible LM-matrix. Note that the determinant of A is a polynomial in $\mathcal{T} = \mathcal{N}(T)$ over $\mathbf{K}$.

THEOREM 3.5. (MUROTA, 1989A) *Let* $A = \binom{Q}{T}$ *be a nonsingular LM-matrix with respect to* $\mathbf{K}$, *and* $\mathcal{T} = \mathcal{N}(T)$.

(1) The determinant $\det A$ *is an irreducible polynomial in the ring* $\mathbf{K}[\mathcal{T}]$ *if* A *is LM-irreducible. Conversely, if* $\det A$ *is an irreducible polynomial, then there exists in the CCF of* A *at most one diagonal block which contains elements of* $\mathcal{T}$ *and all the other diagonal blocks are* 1×1 *matrices over* $\mathbf{K}$.

(2) Each element of $\mathcal{T}$ *appears in* $\det A$ *if* A *is irreducible.*

(3) A^{-1} *is completely dense, i.e.* $(A^{-1})_{ij} \neq 0$, $\forall (i,j)$, *if* A *is irreducible.*

A minor (subdeterminant) of A is also a polynomial in $\mathcal{T} = \mathcal{N}(T)$ over $\mathbf{K}$. Let $d_k(\mathcal{T}) \in \mathbf{K}[\mathcal{T}]$ denote the k-th determinantal divisor of A, i.e. the greatest common divisor of all minors of order k in A as polynomials in $\mathcal{T}$ over $\mathbf{K}$.

THEOREM 3.6. (MUROTA, 1987B, 1989B, 1991) *Let* A *be an irreducible LM-matrix with respect to* $\mathbf{K}$; *put* $R = \mathrm{Row}(A)$, $C = \mathrm{Col}(A)$.

(a) In case $|R| = |C|$ (> 0): $d_k(\mathcal{T}) \in \mathbf{K} - \{0\}$ for $k = 1, \ldots, |R| - 1$.

(b) In case $|R| < |C|$: $d_k(\mathcal{T}) \in \mathbf{K} - \{0\}$ for $k = 1, \ldots, |R|$.

(c) In case $|R| > |C|$: $d_k(\mathcal{T}) \in \mathbf{K} - \{0\}$ for $k = 1, \ldots, |C|$.

Theorems 3.5 and 3.6 together imply the following.

THEOREM 3.7. (MUROTA, 1991) *Let* A *be an LM-matrix of rank* r *with respect to* $\mathbf{K}$. *The decomposition of the* r*-th determinantal divisor* $d_r(\mathcal{T})$ *of* A *into irreducible factors in the ring* $\mathbf{K}[\mathcal{T}]$ *is given by*

$$d_r(\mathcal{T}) = \alpha \cdot \prod_{k=1}^{b} \det \bar{A}[R_k, C_k],$$

where $\bar{A}[R_k, C_k]$ $(k = 1, \ldots, b)$ *are the irreducible square blocks in the CCF of* A, *and* $\alpha \in \mathbf{K} - \{0\}$.

A submatrix $A[I, C]$, where $I \subseteq R$, of an LM-matrix A is again an LM-matrix, for which the CCF is defined. We denote by $\mathcal{P}_{\mathrm{CCF}}(I)$ the partition of C (with a partial order among blocks) in the CCF of $A[I, C]$. In some applications we are concerned with the family of partitions $\{\mathcal{P}_{\mathrm{CCF}}(I) \mid I \in \mathcal{B}\}$, where

$$\mathcal{B} = \{I \subseteq R \mid \mathrm{rank}\, A = \mathrm{rank}\, A[I, C] = |I|\}.$$

A concise characterization to the coarsest common refinement of $\{\mathcal{P}_{\mathrm{CCF}}(I) \mid I \in \mathcal{B}\}$ is given in Murota (1990a).

With an $m \times n$ mixed matrix $A = Q + T$ with respect to $\mathbf{K}$ we associate a $(2m) \times (m+n)$

LM-matrix

$$\tilde{A} = \begin{pmatrix} I_m & Q \\ -\mathrm{diag}\,[t_1, \ldots, t_m] & T \end{pmatrix} = \begin{pmatrix} \tilde{Q} \\ \tilde{T} \end{pmatrix},$$

where $t_1, \ldots, t_m$ are "new" indeterminates (in $\mathbf{F}$). Note that $\operatorname{rank} \tilde{A} = \operatorname{rank} A + m$. Furthermore, the CCF of $\tilde{A}$ yields the finest block-triangular matrix which can be obtained from A by means of the transformation of the form $S\,A\,P_c$ with a nonsingular matrix S over $\mathbf{K}$ (free from $\mathcal{N}(T)$) and a permutation matrix P_c.

3.2. Algorithm for the combinatorial canonical form

An efficient algorithm is described here which computes the CCF of an LM-matrix A of (3.2) in $O(n^3 \log n)$ time with arithmetic operations in the subfield $\mathbf{K}$ only. Note that for an LM-matrix A of (3.2) the Rank Identity (Theorem 3.1) specializes to

$$\operatorname{rank} A = \max\{\operatorname{rank} Q[R_Q, J] + \text{term-rank}\, T[R_T, C - J] \mid J \subseteq C\}. \tag{3.6}$$

In order to illustrate a connection between the CCF and the Dulmage-Mendelsohn decomposition, we first restrict ourselves to a nonsingular LM-matrix A. In this case the CCF can be found as follows:

[Algorithm (outline) for the CCF of a nonsingular A]

1. Find $J \subseteq C$ such that both $Q[R_Q, J]$ and $T[R_T, C - J]$ are nonsingular (such J with $|J| = |R_Q| = |C| - |R_T|$ exists by (3.6)).

2. Let S denote the inverse of $Q[R_Q, J]$ and put

$$A' := \begin{pmatrix} S & O \\ O & I \end{pmatrix} A.$$

3. Find the Dulmage-Mendelsohn decomposition $\bar{A}$ of A', namely, $\bar{A} := P_r\ A'\ P_c$ with suitable permutation matrices P_r and P_c. □

The first step is nothing but the well-studied problem of matroid partition and a number of efficient algorithms are available for it. The Dulmage-Mendelsohn decomposition in step 3 can be computed by first finding a maximum (perfect) matching in the bipartite graph associated with A', i.e. the graph denoted as $G(A')$ at the beginning of section 3.1, and then decomposing an auxiliary digraph into strongly connected components. See Murota (1987b) for more detail on the Dulmage-Mendelsohn decomposition.

For the LM-matrix of Example 3.1, which is nonsingular, we can take $J = \{x_4, x_3, x_5\}$ in step 1. The transformation matrix S given in Example 3.1 is equal to the inverse of

$$Q[R_Q, J] = \begin{array}{c} \begin{array}{ccc} x_4 & x_3 & x_5 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}.$$

For a general (not necessarily nonsingular) LM-matrix it has been shown that the CCF can be constructed by identifying the minimum cuts in an independent-flow problem.

See Murota (1987b) and Murota *et al.* (1987) for this reduction and Fujishige (1991) for independent-flow problems.

The detail of the algorithm for a general LM-matrix A of (3.2) is now described. As before let $R_T = \mathrm{Row}(T)$ and $C = \mathrm{Col}(A)$. Furthermore let C_Q be a disjoint copy of C, where the copy of $j \in C$ will be denoted as $j_Q \in C_Q$. The algorithm works with a graph $G = (V, B)$ with vertex set $V = R_T \cup C_Q \cup C$ and arc set $B = B_T \cup B_C \cup B^+ \cup M$, where

$$B_T = \{(i,j) \mid i \in R_T, j \in C, T_{ij} \neq 0\}, \quad B_C = \{(j_Q, j) \mid j \in C\},$$

and B^+ and M are sets of arcs which are defined and updated in the algorithm; B^+ consists of arcs from C_Q to C_Q and M from C to $R_T \cup C_Q$. The set of end-vertices of M (vertices incident to an arc in M) will be designated as ∂M ($\subseteq V$). The variable P is a two-dimensional array, or a matrix (over $\mathbf{K}$), of size $|R_Q| \times |C|$, where $P = Q$ at the beginning of the algorithm (step 1 below). The variable *base* is a one-dimensional array, or a vector, of size $|R_Q|$, which represents a mapping (correspondence): $R_Q \to C \cup \{0\}$.

[Algorithm for the CCF of a general A]

1. $M := \emptyset$; $\mathit{base}[i] := 0$ $(i \in R_Q)$; $P[i,j] := Q_{ij}$ $(i \in R_Q, j \in C)$.
2. $I := \{i \in C \mid i_Q \in \partial M \cap C_Q\}$;
   $J := \{j \in C - I \mid \text{For all } i,\ \mathit{base}[i] = 0 \text{ implies } P[i,j] = 0\}$;
   $S_T^+ := R_T - \partial M$; $S_Q^+ := \{j_Q \in C_Q \mid j \in C - (I \cup J)\}$; $S^+ := S_T^+ \cup S_Q^+$;
   $S^- := C - \partial M$;
   $B^+ := \{(i_Q, j_Q) \mid h \in R_Q, j \in J, P[h,j] \neq 0, i = \mathit{base}[h]\}$;
   If there exists in G a directed path from S^+ to S^- then go to step 3; otherwise (including the case where $S^+ = \emptyset$ or $S^- = \emptyset$) go to step 4.
3. Let L ($\subseteq B$) be (the set of arcs on) a shortest path from S^+ to S^-;
   $M := (M - L) \cup \{(j,i) \mid (i,j) \in L \cap B_T\} \cup \{(j, j_Q) \mid (j_Q, j) \in L \cap B_C\}$;
   If the initial vertex ($\in S^+$) of the path L belongs to S_Q^+, then do the following:
   {Let j_Q ($\in S_Q^+ \subseteq C_Q$) be the initial vertex;
   Find h such that $\mathit{base}[h] = 0$ and $P[h,j] \neq 0$;
   $\mathit{base}[h] := j$; $w := 1/P[h,j]$;
   $P[k,l] := P[k,l] - w \times P[k,j] \times P[h,l]$ $(h \neq k \in R_Q, l \in C)$ };
   For all $(i_Q, j_Q) \in L \cap B^+$ (in the order from S^+ to S^- along L) do the following:
   {Find h such that $i = \mathit{base}[h]$;
   $\mathit{base}[h] := j$; $w := 1/P[h,j]$;
   $P[k,l] := P[k,l] - w \times P[k,j] \times P[h,l]$ $(h \neq k \in R_Q, l \in C)$ };
   Go to step 2.
4. Let V_∞ ($\subseteq V$) be the set of vertices reachable from S^+ by a directed path in G;
   Let V_0 ($\subseteq V$) be the set of vertices reachable to S^- by a directed path in G;
   $C_0 := C \cap V_0$; $C_\infty := C \cap V_\infty$;
   Let G' denote the graph obtained from G by deleting the vertices $V_0 \cup V_\infty$ (and arcs incident thereto);
   Decompose G' into strongly connected components $\{V_\lambda \mid \lambda \in \Lambda\}$ ($V_\lambda \subseteq V$);
   Let $\{C_k \mid k = 1, \ldots, b\}$ be the subcollection of $\{C \cap V_\lambda \mid \lambda \in \Lambda\}$ consisting of all the nonempty sets $C \cap V_\lambda$, where C_k's are indexed in such a way that for $l < k$ there does not exist a directed path in G' from C_k to C_l;
   $R_0 := (R_T \cap V_0) \cup \{h \in R_Q \mid \mathit{base}[h] \in C_0\}$;
   $R_\infty := (R_T \cap V_\infty) \cup \{h \in R_Q \mid \mathit{base}[h] \in C_\infty \cup \{0\}\}$;

$R_k := (R_T \cap V_k) \cup \{h \in R_Q \mid base[h] \in C_k\}$ $(k = 1, \ldots, b)$;

$\bar{A} := P_r \begin{pmatrix} P \\ T \end{pmatrix} P_c$, where the permutation matrices P_r and P_c are determined so that the rows and the columns of $\bar{A}$ are ordered as $(R_0; R_1, \ldots, R_b; R_\infty)$ and $(C_0; C_1, \ldots, C_b; C_\infty)$, respectively. □

At each execution of step 3 the size of M increases by one, and at the termination of the algorithm we have the relation: rank $A = |M|$. The matrix $\bar{A}$ is the CCF of the input matrix A, where $\{R_0; R_1, \ldots, R_b; R_\infty\}$ and $\{C_0; C_1, \ldots, C_b; C_\infty\}$ give the partitions of the row set and the column set, respectively.

The shortest path in step 3 and the strongly connected components in step 4 can be found in time linear in the size of the graph G, by means of the standard graph algorithms (see Aho *et al.*, 1974).

Note also that the updates of P in step 3 are the standard pivoting operations on P, which is a matrix over the subfield $\mathbf{K}$. The sparsity of P should be taken into account in actual implementations; for example, $P[h, j] = 0$ if $base[h] = 0$ and $j \in I \cup J$.

When the transformation matrix S in (3.5) is to be computed, we introduce another two-dimensional array, or matrix (over $\mathbf{K}$), of size $|R_Q| \times |R_Q|$, set $S := I$ (identity matrix) in step 1, and update S together with P in step 3 according to the formula:

$$S[k, l] := S[k, l] - w \times P[k, j] \times S[h, l] \quad (h \neq k \in R_Q, l \in R_Q).$$

Example 3.2. The algorithm above is illustrated here for a 4×5 LM-matrix $A = \binom{Q}{T}$ of (3.2) with

$$Q = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{pmatrix} \end{array}, \quad T = \begin{array}{c} \\ \begin{matrix} f_1 \\ f_2 \end{matrix} \end{array} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{pmatrix} t_1 & 0 & 0 & 0 & t_2 \\ 0 & t_3 & 0 & 0 & t_4 \end{pmatrix} \end{array},$$

where $\mathrm{Col}(A) = C = \{x_1, x_2, x_3, x_4, x_5\}$ and $\mathrm{Row}(T) = R_T = \{f_1, f_2\}$. We work with a 2×5 matrix P, a 2×2 matrix S, and a vector *base* of size 2. The copy of C is denoted as $C_Q = \{x_{1Q}, x_{2Q}, x_{3Q}, x_{4Q}, x_{5Q}\}$.

The flow of computation is traced below.

1. $M := \emptyset$;

$$base := \begin{array}{c} \begin{matrix} r_1 \\ r_2 \end{matrix} \end{array} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad P := \begin{array}{c} \\ \begin{matrix} r_1 \\ r_2 \end{matrix} \end{array} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{pmatrix} \end{array}, \quad S := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2. $I := \emptyset$; $J := \{x_5\}$;

$S_T^+ := \{f_1, f_2\}$; $S_Q^+ := \{x_{1Q}, x_{2Q}, x_{3Q}, x_{4Q}\}$; $S^+ := \{f_1, f_2, x_{1Q}, x_{2Q}, x_{3Q}, x_{4Q}\}$;

$S^- := \{x_1, x_2, x_3, x_4, x_5\}$;

$B^+ := \emptyset$;

There exists a path from S^+ to S^-. [See $G^{(0)}$ in fig. 3]

3. $L := \{(x_{1Q}, x_1)\}$; $M := \{(x_1, x_{1Q})\}$;

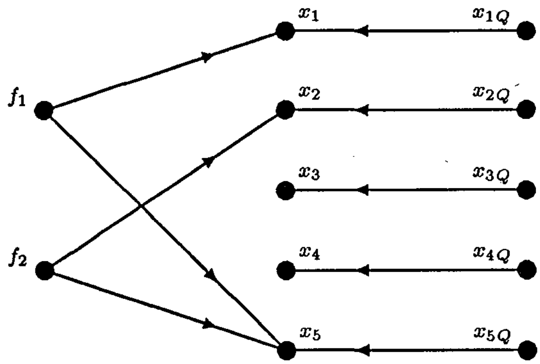

Figure 3. Graph $G^{(0)}$ of Example 3.2.

The initial vertex x_{1Q} of L is in S_Q^+, and the matrices are updated (with $h = r_1$) to

$$base := \begin{array}{c} r_1 \\ r_2 \end{array} \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \quad P := \begin{array}{c} \\ r_1 \\ r_2 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{pmatrix} \end{array}, \quad S := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Noting $L \cap B^+ = \emptyset$ we return to step 2.

2. $I := \{x_1\}$; $J := \{x_5\}$;
$S_T^+ := \{f_1, f_2\}$; $S_Q^+ := \{x_{2Q}, x_{3Q}, x_{4Q}\}$; $S^+ := \{f_1, f_2, x_{2Q}, x_{3Q}, x_{4Q}\}$;
$S^- := \{x_2, x_3, x_4, x_5\}$;
$B^+ := \emptyset$;
There exists a path from S^+ to S^-. [See $G^{(1)}$ in fig. 4]

3. $L := \{(x_{2Q}, x_2)\}$; $M := \{(x_1, x_{1Q}), (x_2, x_{2Q})\}$;
The initial vertex x_{2Q} of L is in S_Q^+, and the matrices are updated (with $h = r_2$) to

$$base := \begin{array}{c} r_1 \\ r_2 \end{array} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad P := \begin{array}{c} \\ r_1 \\ r_2 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{pmatrix} \end{array}, \quad S := \begin{pmatrix} 1 & -1/2 \\ 0 & 1 \end{pmatrix}.$$

Noting $L \cap B^+ = \emptyset$ we return to step 2.

2. $I := \{x_1, x_2\}$;
$J := \{x_3, x_4, x_5\}$;
$S_T^+ := \{f_1, f_2\}$; $S_Q^+ := \emptyset$; $S^+ := \{f_1, f_2\}$; $S^- := \{x_3, x_4, x_5\}$;
$B^+ := \{(x_{1Q}, x_{3Q}), (x_{1Q}, x_{4Q}), (x_{2Q}, x_{3Q}), (x_{2Q}, x_{4Q})\}$;
There exists a path from S^+ to S^-. [See $G^{(2)}$ in fig. 5]

3. $L := \{(f_1, x_5)\}$; $M := \{(x_1, x_{1Q}), (x_2, x_{2Q}), (x_5, f_1)\}$;
The initial vertex $f_1 \notin S_Q^+$ and $L \cap B^+ = \emptyset$. Therefore the matrices remain unchanged and we return to step 2.

2. $I := \{x_1, x_2\}$; $J := \{x_3, x_4, x_5\}$;
$S_T^+ := \{f_2\}$; $S_Q^+ := \emptyset$; $S^+ := \{f_2\}$; $S^- := \{x_3, x_4\}$;
$B^+ := \{(x_{1Q}, x_{3Q}), (x_{1Q}, x_{4Q}), (x_{2Q}, x_{3Q}), (x_{2Q}, x_{4Q})\}$;

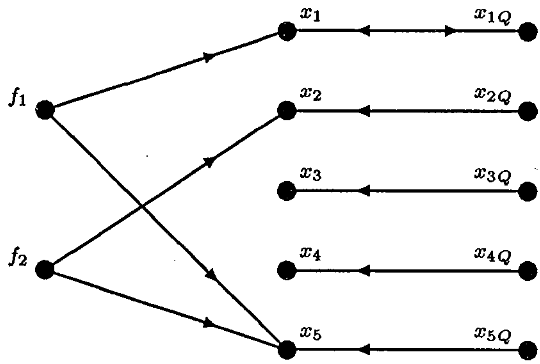

Figure 4. Graph $G^{(1)}$ of Example 3.2.

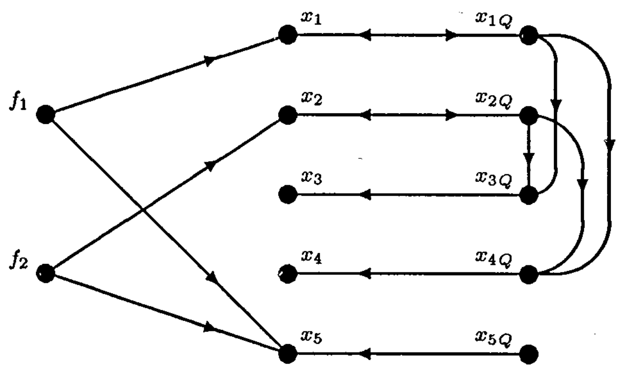

Figure 5. Graph $G^{(2)}$ of Example 3.2.

There exists a path from S^+ to S^-. [See $G^{(3)}$ in fig. 6]

3. $L := \{(f_2, x_2), (x_2, x_{2Q}), (x_{2Q}, x_{3Q}), (x_{3Q}, x_3)\}$;
$M := \{(x_1, x_{1Q}), (x_3, x_{3Q}), (x_5, f_1), (x_2, f_2)\}$;
The initial vertex $f_2 \notin S_Q^+$ and $L \cap B^+ = \{(x_{2Q}, x_{3Q})\}$. The matrices are updated (with $h = r_2$) to

$$base := \begin{array}{c} r_1 \\ r_2 \end{array} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}, \quad P := \begin{array}{c} \\ r_1 \\ r_2 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{pmatrix} \end{array}, \quad S := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

2. $I := \{x_1, x_3\}$; $J := \{x_2, x_4, x_5\}$;
$S_T^+ := \emptyset$; $S_Q^+ := \emptyset$; $S^+ := \emptyset$; $S^- := \{x_4\}$;
$B^+ := \{(x_{1Q}, x_{2Q}), (x_{3Q}, x_{2Q}), (x_{3Q}, x_{4Q})\}$;

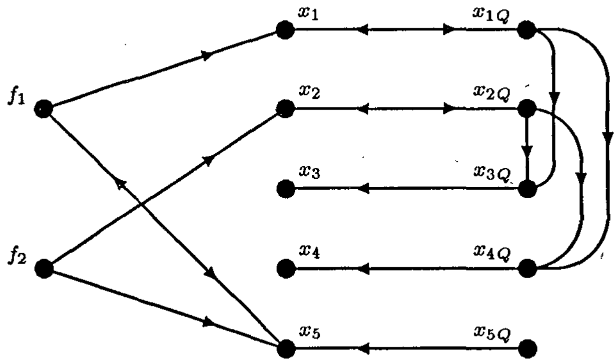

Figure 6. Graph $G^{(3)}$ of Example 3.2.

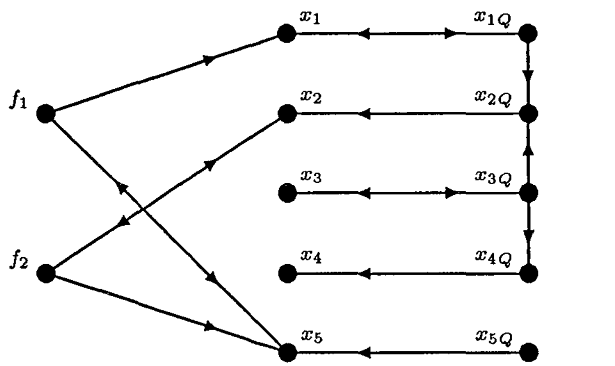

Figure 7. Graph $G^{(4)}$ of Example 3.2.

There exists no path from S^+ $(= \emptyset)$ to S^-. [See $G^{(4)}$ in fig. 7]

4. $V_\infty := \emptyset$; $V_0 := \{x_3, x_4, x_{3Q}, x_{4Q}\}$;
$C_0 := \{x_3, x_4\}$; $C_\infty := \emptyset$;
Strongly connected components of G' are given by $\{V_{\lambda_1}, V_{\lambda_2}\}$, where
$V_{\lambda_1} = \{x_1, x_2, x_5, x_{1Q}, x_{2Q}, f_1, f_2\}$ and $V_{\lambda_2} = \{x_{5Q}\}$;
Since $C \cap V_{\lambda_2} = \emptyset$, we have $b := 1$ and $C_1 := C \cap V_{\lambda_1} = \{x_1, x_2, x_5\}$;
$R_0 := \{r_2\}$; $R_\infty := \emptyset$; $R_1 := \{r_1, f_1, f_2\}$;

$$\bar{A} := P_{\mathrm{r}} \begin{pmatrix} P \\ T \end{pmatrix} P_{\mathrm{c}} = \begin{array}{c} \\ r_2 \\ r_1 \\ f_1 \\ f_2 \end{array} \begin{array}{c} \begin{array}{ccccc} x_3 & x_4 & x_1 & x_2 & x_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 2 & 0 \\ & & 1 & -1 & 0 \\ & & t_1 & 0 & t_2 \\ & & 0 & t_3 & t_4 \end{pmatrix} \end{array}$$

is the CCF. □

4. Some Properties of Dynamical Systems

In this section we shall briefly mention some control-theoretic problems (e.g. Rosenbrock, 1970; Wolovich, 1974; Wonham, 1979) which have been successfully treated using the mathematical model introduced in section 2. It should be understood that we treat generic (or structural) properties with respect to the parameters $\mathcal{T}$; for example, by "controllability" we mean the generic (or structural) controllability. Though not emphasized below, it should be noted that all the combinatorial characterizations lead to practically efficient algorithms which run in polynomial time and which are composed of graph manipulations and arithmetic operations on rational numbers.

Throughout this section, $D(s)$ denotes a (square or nonsquare) polynomial matrix expressed as (2.6) with (A1) and (A2). This implies, in particular, that $D(s)$ is a mixed matrix with respect to $\mathbf{K} = \mathbf{Q}(s)$. The entries of $D(s)$ are not restricted to linear functions in s.

4.1. Dynamical degree, model matching and disturbance decoupling

The degrees of minors of $D(s)$ are often of system-theoretic interest. For example, for the descriptor system (2.2), the degree of $\det(A - sF)$ in s, i.e.

$$\delta(A - sF) = \deg_s \det(A - sF),$$

is one of the fundamental characteristics, sometimes called the *dynamical degree.* It expresses the number of exponential modes, or the number of state-space variables when (2.2) is reduced to the standard state-space (2.1).

Let $R = \mathrm{Row}(D)$ and $C = \mathrm{Col}(D)$. We define

$$\delta(D) = \deg_s \det D(s),$$

and furthermore, for $I_0 \subseteq R$, $J_0 \subseteq C$ and $k \geq \max(|I_0|, |J_0|)$ we define

$$\delta_k(D; I_0, J_0) = \max\{\delta(D[I,J]) \mid I \supseteq I_0,\ J \supseteq J_0, |I| = |J| = k\}.$$

The problem of computing $\delta(D)$ (or the dynamical degree if $D = A - sF$) has been formulated as a weighted matroid-partition/intersection problem (or independent matching problem) and an efficient algorithm for $\delta(D)$ has been given in Murota (1985); see also Murota (1987b, 1989b). Recently, this result has been extended by Murota and van der Woude (1991) to compute $\delta_k(D; I_0, J_0)$. This extension has made it possible to compute the structure at infinity of transfer matrices, to test for the solvability of the exact model matching problem, and to test for the solvability of the disturbance decoupling problem. See Murota and van der Woude (1991) for details.

4.2. Smith normal form and controllability/observability

The Smith normal form of $m \times n$ polynomial matrix $D(s)$ is often of system-theoretic interest. For example, the controllability (of the exponential modes) of the descriptor system (2.2) (with $A - sF$ nonsingular) is known to be equivalent to the condition that the Smith normal form of $D(s) = (A - sF \mid B)$ is equal to $(I_m \mid O)$. This is also equivalent to saying that

$$d_m(s) = 1,$$

where $d_m(s)$ denotes the monic greatest common divisor (in $\mathbf{F}[s]$) of all the $m \times m$ minors of $D(s)$.

It has been shown in Murota (1987a) that $\deg d_m(s)$ can be computed by solving a weighted-matroid union problem; based on this characterization an efficient algorithm has been constructed in Murota (1987a, 1987b) for testing for the controllability.

A recent paper (Murota, 1991) shows that the Smith normal form of $D(s)$ has a very simple structure, as stated below, and hence it can be computed efficiently by solving a weighted matroid-partition/intersection problem. The CCF plays the primary role in deriving this result.

THEOREM 4.1. (Murota, 1991) *Assume (A1) and (A2) for $D(s)$ of (2.6), and let*

$$\mathrm{diag}\,[\epsilon_1(s), \ldots, \epsilon_r(s), 0, \ldots, 0]$$

be the Smith normal form of $D(s)$, where r is the rank of $D(s)$. Then,

$$\epsilon_k(s) = s^{p_k}, \quad k = 1, \ldots, r-1,$$

for some $p_1 \leq \cdots \leq p_{r-1}$.

4.3. Fixed modes

Let $D(s)$ be a nonsingular matrix expressed as (2.6) with

$$Q_D(s) = Q^0 + sQ^1, \quad T_D(s) = (T^0 + sT^1) + \hat{K},$$

where (A1) and (A2) are assumed again. We distinguish $\mathcal{N}(\hat{K})$ from $\mathcal{N}(T^0) \cup \mathcal{N}(T^1)$, regarding the latter as the parameters of fixed values describing a given system and the former as the parameters that we can control or design.

The *fixed polynomial* $\psi(s)$ is defined as the greatest common divisor of the set of $\det D(s)$ when $\hat{K}$ runs over all admissible matrices, i.e.

$$\psi(s) = \gcd\{\det D(s) \mid \hat{K} \in \mathcal{K}\},$$

where $\mathcal{K}$ denotes the set of all real matrices of the given zero/nonzero pattern. A complex number $\lambda \in \mathbf{C}$ is called a *fixed mode* if $\psi(\lambda) = 0$.

When the state-space (2.1), augmented by $\mathbf{y}(t) = C\mathbf{x}(t)$, describe a decentralized control system with ν local control stations, the local nondynamic output feedback $\mathbf{u}(t) = K\mathbf{y}(t)$ is specified by a block-diagonal real matrix

$$K = \text{block-diag}\,[K_1, \ldots, K_\nu],$$

where K_i represents the output feedback at the ith control station ($i = 1, \ldots, \nu$). The concept of fixed modes defined above agrees with the usual definition if we define

$$D(s) = \begin{pmatrix} A - sF & B & O \\ O & -I & K \\ C & O & -I \end{pmatrix},$$

$$Q^0 = \begin{pmatrix} Q_A & Q_B & O \\ O & -I & Q_K \\ Q_C & O & -I \end{pmatrix}, \quad Q^1 = \begin{pmatrix} -Q_F & O & O \\ O & O & O \\ O & O & O \end{pmatrix},$$

$$T^0 = \begin{pmatrix} T_A & T_B & O \\ O & O & O \\ T_C & O & O \end{pmatrix}, \quad T^1 = \begin{pmatrix} -T_F & O & O \\ O & O & O \\ O & O & O \end{pmatrix},$$

$$\hat{K} = \begin{pmatrix} O & O & O \\ O & O & T_K \\ O & O & O \end{pmatrix},$$

where the admissible feedback structure is assumed to be specified by the mixed matrix $K = Q_K + T_K$ with $\mathcal{N}(T_K)$ representing the free parameters.

It has been shown in Murota (1989c) that the fixed polynomial can be identified with the aid of Theorem 3.5, and an efficient algorithm for computing $\deg_s \psi(s)$ has been given. Note that the CCF plays the primary role again.

5. Conclusion

The mathematical framework presented in this chapter for the analysis of dynamical systems shares a lot in common with the matroid-theoretic methods developed for other engineering problems. The reader is referred to Iri (1983), Recski (1989) and Sugihara (1986).

Though a considerable number of papers have been published on "nice" applications of matroid theory, they still remain "nice" theoretical applications. It must be admitted that those matroid-theoretic methods have never been tested against real-world problems in industry. It is hoped that this short survey, and in particular, the detailed description of the algorithm in section 3.2, will contribute to the integration of theory and practice.

References

M. Aigner (1979), *Combinatorial Theory*, Springer-Verlag, Berlin.

A.V. Aho, J.E. Hopcroft and J.D. Ullman (1974), *The Design and Analysis of Computer Algorithms*, Addison-Wesley, Reading, MA.

A.L. Dulmage and N.S. Mendelsohn (1959), "A structure theory of bipartite graphs of finite exterior dimension", *Trans. Royal Soc. Canada*, Section III, **53**, 1–13.

S. Fujishige (1991), *Submodular Functions and Optimization*, Annals of Discrete Math., **47**, North-Holland, Amsterdam.

M. Iri (1983), "Applications of matroid theory", *Mathematical Programming – The State of the Art*, A. Bachem, M. Grötschel and B. Korte, eds., Springer-Verlag, Berlin, 158–201.

R.E. Kalman (1963), "Mathematical description of linear dynamical systems", *SIAM J. Control, Ser. A*, **1**, 152–192.

C.-T. Lin (1974), "Structural controllability", *IEEE Trans. Auto. Control*, **AC-19**, 201–208.

D.G. Luenberger (1977), "Dynamic equations in descriptor form", *IEEE Trans. Auto. Control*, **AC-22**, 312–321.

K. Murota (1985), "Use of the concept of physical dimensions in the structural approach to systems analysis", *Japan J. Appl. Math.*, **2**, 471–494.

K. Murota (1987a), "Refined study on structural controllability of descriptor systems by means of matroids", *SIAM J. Control and Optimization*, **25**, 967–989.

K. Murota (1987b), *Systems Analysis by Graphs and Matroids – Structural Solvability and Controllability*, Algorithms and Combinatorics, **3**, Springer-Verlag, Berlin-Heidelberg.

K. Murota (1989a), "On the irreducibility of layered mixed matrices", *Linear and Multilinear Algebra*, **24**, 273–288.

K. Murota (1989b), "Some recent results in combinatorial approaches to dynamical systems", *Linear Algebra and Its Applications*, **122/123/124**, 725–759.

K. Murota (1989c), "A matroid-theoretic approach to structurally fixed modes of control systems", *SIAM J. Control and Optimization*, **27**, 1381–1402.

K. Murota (1990a), "Principal structure of layered mixed matrices", *Discrete Appl. Math.*, **27**, 221–234.

K. Murota (1990b), "Combinatorial canonical form of mixed matrix and its application" (in Japanese), *Proc. Annual Symp. Math. Soc. Japan, Div. Appl. Math.*, Okayama, 222–241.

K. Murota (1990c), *Hierarchical Decomposition of Symmetric Discrete Systems by Matroid and Group Theories*, Technical Report METR 90-20, Dept. of Math. Eng. and Info. Phy., University of Tokyo. To appear in *Math. Prog. Ser. A*.

K. Murota (1991), "On the Smith normal form of structured polynomial matrices", *SIAM J. Matrix Anal. Appl.*, **12**(4), 747–765.

K. Murota (1992), "Mixed matrices – irreducibility and decomposition", *Combinatorial and Graph-Theoretical Problems in Linear Algebra*, R.A. Brualdi, S. Friedland and V. Klee, eds., The IMA Volumes in Math. and its Applications, Springer-Verlag, Berlin, to appear.

K. Murota and M. Iri (1985), "Structural solvability of systems of equations — a mathematical formulation for distinguishing accurate and inaccurate numbers in structural analysis of systems", *Japan J. Appl. Math.*, **2**, 247–271.

K. Murota, M. Iri and M. Nakamura (1987), "Combinatorial canonical form of layered mixed matrices and its application to block-triangularization of systems of equations", *SIAM J. Algebraic and Discrete Methods*, **8**, 123–149.

K. Murota and J. van der Woude (1991), "Structure at infinity of structured descriptor systems and its applications", *SIAM J. Control and Optimization*, **29**, 878–894.

A. Recski (1989), *Matroid Theory and Its Applications in Electric Network Theory and in Statics*, Springer-Verlag, Berlin-Heidelberg.

H.H. Rosenbrock (1970), *State-space and Multivariable Theory*, Nelson, London.

K. Sugihara (1986), *Machine Interpretation of Line Drawings*, MIT Press, Cambridge, MA.

W.A. Wolovich (1974), *Linear Multivariable Systems*, Springer-Verlag, NY.

W.M. Wonham (1979), *Linear Multivariable Control: A Geometric Approach*, Springer-Verlag, NY.