Please read the following instructions first.

○ This is an open book exam. You may use the books and notes you bring in. Any reference via the Internet, electronic communication, or discussion/collaboration with anyone else is not allowed.

○ Each individual selects four out of the five problems as the primary ones for the exam grading. The additional problem, if you wish to submit as well, will not get a score higher than the average of the primary ones.

○ The elaborated items and associated score points in each problem are to help you organize your thoughts and estimate your gain/loss in the total score. Do not take the sub-items as independent questions.

A remark about the point distribution. Relatively easier issues have relatively more rewarding points.

○ Try to make your answers and arguments as concise as you can. Irrelevant comments are subject to point deduction as incorrect arguments are.

This page is used as the cover page upon submission.

Your first name in print:

Your last name in print:

Your choice of the four primary problems for grading:

Signature:

Your decision and effort in taking this exam are appreciated, regardless of the outcome. Good luck!
1. Direct methods for solving a system of linear equations

Consider numerical solution of a system of linear equations

\[ Ax = b, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \text{span}(A). \]

a. (5) Describe a solution method when \( m = n \) and \( A \) is triangular (via proper permutations in the equations and unknowns ) with non-zero diagonal elements.

b. (5) Describe a solution method when \( A \) is orthogonal.

c. (10) Describe the direct method via the QR factorization when \( A \) is of full column rank.

d. (5) Extend the method in c to find a least square solution when \( b \) is not necessarily in the column space of \( A \) while \( A \) remains full rank in columns.

2. Iterative solutions of linear and nonlinear equations

a. (5) Describe the fixed point method in its generic form for solving a system of (linear or nonlinear) equations

\[ F(x) = 0, \quad x \in \mathbb{R}^n, \quad F : \mathbb{R}^n \rightarrow \mathbb{R}^n, \]

and describe the necessary condition for convergence.

b. (10) Describe a particular fixed point method for solving a linear system \( F(x) = Ax - b \) where \( A \) is symmetric and positive definite.

c. (10) Point out some fundamental differences (at least two) in algorithm properties between the conjugate gradient (CG) method and the method in Item b. for solving a symmetric, positive definite linear system \( Ax = b \).
3. Error propagation and stability analysis

Consider the evaluation of the following three-terms recurrence (or difference) equation,

\[ y_{k+1} = \alpha \cdot y_k + \beta \cdot y_{k-1}, \quad k = 1, 2, 3, \cdots \]

provided with the initial values \( y_0 \) and \( y_1 \), where \( \alpha \) and \( \beta \) are constant coefficients.

Suppose that the initial values are rendered as \( \tilde{y}_j, \tilde{y}_j = y_i (1 + \eta_j) \), with relative errors \( \eta_j \), \( j = 0, 1 \). Assume for simplicity that there is no further error introduced in the evaluation.

a. (7) Consider first the special case with \( \beta = 0 \). Find a closed-form expression of \( y_k, k \geq 2 \), in terms of \( \alpha \) and \( \tilde{y}_1 \). Describe the absolute and relative errors in \( y_k \), respectively.

b. (7) Find a closed-form expression of \( y_k, k \geq 2 \), for the case \( \beta \neq 0 \), in terms of the constant coefficients and the perturbed initial values. (Hint : it may be helpful to use matrix expressions.)

c. (7) Describe the absolute and relative errors in \( y_k, k \geq 2 \), in terms of the constant coefficients, the initial values and the initial errors.

d. (4) Consider the specific case that \( \alpha = 2.25, \beta = -0.5 \). Describe an unstable situation where the absolute and relative errors in \( y_k \) grow rapidly with \( k \).
4. Discretization and solution of PDEs

Consider the two dimensional heat equation, provided with the Dirichlet boundary condition and the initial condition,

\[ \frac{\partial u(x, y, t)}{\partial t} = p \frac{\partial^2 u(x, y, t)}{\partial x^2} + q \frac{\partial^2 u(x, y, t)}{\partial y^2}; \quad (x, y) \in \Omega = (a, b) \times (a, b), \]

subject to

\[ u(x, y, 0) = h(x, y), \quad (x, y) \in \Omega \]
\[ u(x, y, t) = 0, \quad (x, y) \in \partial \Omega, \quad t \in [0, T] \]

where \( p \) and \( q \) are positive constants, \( b > a \), \( T > 0 \), and \( \partial \Omega \) denotes the boundary of \( \Omega \).

a. (7) Describe a scheme of discretizing the spatial variables \((x, y)\) and the temporal variable \( t \).

b. (8) Describe the discretized unknowns and the system of linear equations in the unknowns. Describe the system size and structure, based on the discretization scheme.

c. (10) Describe a numerical method for solving the system of discretized heat equations.
5. Dimension reduction: theory and applications

a. (20) Provide a clear description of a particular dimension reduction method for a specific application problem, including the assumption(s) or conditions, the basic algorithm, and the main underlying concepts.

b. (5) Point out an alternative method for the same application problem. Comment on the choice between the two methods.