Problem 1: Regular Languages
Fix constant integers $a, b > 0$, and binary strings $u$ and $w$.

(1a) Let $L$ be the set of strings $\{0^{an}1^{bm}w | n > 0, m > 0\}$. Show $L$ is a regular language.

(1b) Let $L'$ be the set of strings $\{0^{an}1^{bn}w | n > 0\}$. Show $L'$ is not regular. Hint: use the pumping lemma.

(1c) Show that if a Turing Machine accepts language $L'$, then it requires $\Omega(\log n)$ space.
Problem 2: Decidability
Suppose $L$ is a language over alphabet $\Sigma$ that is recognized by Turing Machine $M$. Also suppose there is a recursive reduction from $L$ to $\{w | w \in \Sigma^* \land w \not\in L\}$. Show that $L$ is decidable.
Problem 3: Undecidability
Consider the problem: given a 2-tape Turing Machine: does it ever write a non-blank symbol on the second tape? Show this problem is undecidable. Hint: Give a reduction from the Halting Problem for 1-headed Turing Machines.
Problem 4: NP Completeness

A clique of a graph is a set of vertices all pairs of which have an edge between them. The CLIQUE problem is given a graph $G$ a number $k$, determine if there is a clique of $k$ vertices in the graph $G$. The HALF-CLIQUE problem is given a graph $G$ with an even number of vertices, does there exist a clique of $G$ consisting of exactly half the nodes of $G$? Show the HALF-CLIQUE problem is NP-complete. Hint: You can assume the CLIQUE problem is NP-complete.
Problem 5: TM Speedup

Show that for any constant $c > 1$, given deterministic Turing Machine $M$ which on input of length $n$ has time bound $T(n)$, there is another faster Turing Machine $M'$ with time bound $O(n) + T(n)/c$ which accepts the same language as $M$.

(5a) First describe precisely the Turing Machine $M$, including its sets of input symbols $\Sigma$, tape symbols and transition function.

(5b) Then describe precisely the Turing Machine $M'$, including its set of input symbols (also $\Sigma$), tape symbols and transition function. Hint: implement the speed-up by careful design of the new tape symbols and transition function.

(5c) Give a careful proof of the simulation of $M$ by $M'$. 
Problem 6: Restricted Post Correspondence Problems

The \textit{Post Correspondence Problem (PCP)} problem is given alphabet $\Sigma$ and two lists $A$ and $B$ of strings over $\Sigma$, where $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, find $i_1, i_2, \ldots, i_k$ such that $a_{i_1} a_{i_2} \ldots a_{i_k} = b_{i_1} b_{i_2} \ldots b_{i_k}$. You can assume the PCP problem as just stated is undecidable.

(5a) The \textit{binary PCP} problem is the PCP problem restricted to where $\Sigma=\{0,1\}$. Show the binary PCP problem is undecidable for binary strings. Hint: Give a recursive reduction from the PCP problem to the binary PCP problem.

(5b) The \textit{unary PCP} problem is the PCP problem restricted to where $\Sigma=\{1\}$. Show the unary PCP problem is decidable. Hint: You can assume the problem of solving systems of linear equations over the integers is decidable.
Problem 7: K-complexity

Let the K-complexity of a recursive language $L$ be the minimum number of bits required to represent a Turing Machine that accepts the language $L$. Show that it is undecidable given a Turing Machine $M$, to determine the K-complexity of the language of strings accepted by $M$. Hint: The Emptiness problem (is language of a Turing Machine empty?) is undecidable. Give a recursive reduction from the Emptiness problem for Turing Machines to the problem of determining the K-complexity.